Technische Universität Kaiserslautern

Multivariate Anisotropic Periodic Wavelets

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Introduction

Periodic wavelets were first defined for the univariate case [PT95]

- based on shifts by $2\pi/N$, $N \in \mathbb{N}$
- de la Vallée Poussin type wavelets and an FWT [Se98]

For the multivariate generalization

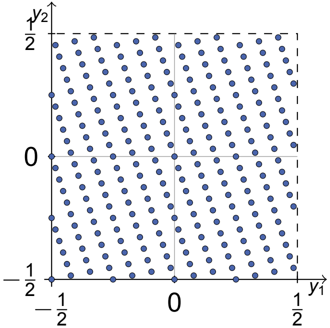
- based on certain shifts by $\mathbf{y} \in \mathbb{T}^d := [-\pi, \pi)^d$
- scaling factor j replaced by a matrix $\mathbf{J}$ [MS03]
- for fixed $|\det \mathbf{J}| = 2$: several matrices $\mathbf{J}$ available [LP10]

$\Rightarrow$ preference of direction

Topics for this talk

- construction of multivariate de la Vallée Poussin type wavelets
- a way to characterize directions

Pattern and Generating Set



The pattern $\mathcal{P}(\mathbf{M})$,

$$\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix}$$

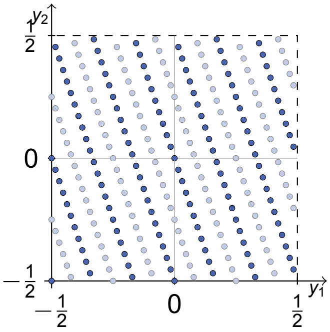
Throughout this talk, let $\mathbf{M} \in \mathbb{Z}^{d \times d}$ be regular.

- pattern
  $\mathcal{P}(\mathbf{M}) := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$
- generating set
  $\mathcal{G}(\mathbf{M}) := \mathbf{M}\mathcal{P}(\mathbf{M}) = \mathbf{M}\left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbb{Z}^d$

We have

- $m := |\mathcal{P}(\mathbf{M})| = |\mathcal{G}(\mathbf{M})| = |\det \mathbf{M}|$
- the group $\left(\mathcal{P}(\mathbf{M}), + \bmod 1\right)$
- subpatterns $\mathcal{P}(\mathbf{N})$, for
  $\mathbf{M} = \mathbf{J}\mathbf{N}, \quad \mathbf{J}, \mathbf{N} \in \mathbb{Z}^{d \times d}$

Pattern and Generating Set



subpattern $\mathcal{P}(\mathbf{N}) \subset \mathcal{P}(\mathbf{M})$,

$$\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix} = \mathbf{J}_Y \begin{pmatrix} 28 & -12 \\ 6 & 2 \end{pmatrix}$$

$$\mathbf{J}_Y := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Throughout this talk, let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be regular.

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  $\mathcal{P}(\mathbf{M}) := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$
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Fourier Basics

For fixed orderings of $\mathcal{G}(\mathbf{M}^\mathrm{T})$ and $\mathcal{P}(\mathbf{M})$:

- Fourier matrix: $\mathcal{F}(\mathbf{M}) := \frac{1}{\sqrt{m}}\left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^\mathrm{T}\mathbf{y}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^\mathrm{T}),\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^{m\times m}$
- The discrete Fourier transform for $\mathbf{a} = (a_\mathbf{y})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^m$ is defined as

$$\hat{\mathbf{a}} = (\hat{a}_\mathbf{h})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^\mathrm{T})} := \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a} \in \mathbb{C}^m.$$

- Fourier coefficients of $f \in L_2(\mathbb{T}^d)$ are given by

$$c_\mathbf{k}(f) := \langle f, \mathrm{e}^{\mathrm{i}\mathbf{k}^\mathrm{T}\circ}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}^\mathrm{T}\mathbf{x}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{k}\in\mathbb{Z}^d.$$

The translation invariant space of $\xi \in L_2(\mathbb{T}^d)$ w.r.t $\mathcal{P}(\mathbf{M})$ is given by

- $V_\mathbf{M}^\xi := \left\{ f;\ f = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{f,\mathbf{y}}\xi(\circ - 2\pi\mathbf{y}), \quad \mathbf{a}_f = (a_{f,\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^m \right\}$
- expressed in Fourier coefficients: For all $\mathbf{h} \in \mathcal{G}(\mathbf{M}^\mathrm{T})$, $\mathbf{z} \in \mathbb{Z}^d$

$$c_{\mathbf{h}+\mathbf{M}^\mathrm{T}\mathbf{z}}(f) = \hat{a}_{f,\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^\mathrm{T}\mathbf{z}}(\xi), \quad \text{where } \hat{\mathbf{a}}_f = \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a}_f.$$

Wavelet Transform

Factorize $\mathbf{M} = \mathbf{JN}$, $|\det \mathbf{J}| = 2$ and take functions $\xi, \varphi \in L_2(\mathbb{T}^d)$ such that

- $\xi(\circ - 2\pi\mathbf{y}),\ \mathbf{y} \in \mathcal{P}(\mathbf{M})$, linear independent
- $\varphi(\circ - 2\pi\mathbf{x}),\ \mathbf{x} \in \mathcal{P}(\mathbf{N})$, linear independent
- $\varphi \in V_{\mathbf{M}}^{\xi}$, i.e. $V_{\mathbf{N}}^{\varphi} \subset V_{\mathbf{M}}^{\xi}$ and hence $\dim V_{\mathbf{N}}^{\varphi} = \frac{1}{2} \dim V_{\mathbf{M}}^{\xi} = \frac{m}{2}$

$\Rightarrow$ $\exists$ wavelet $\psi \in L_2(\mathbb{T}^d)$ s.t. $V_{\mathbf{M}}^{\xi} = V_{\mathbf{N}}^{\varphi} \oplus V_{\mathbf{N}}^{\psi}$.

Decompose $f \in V_{\mathbf{M}}^{\xi}$ into

$$f = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{f,\mathbf{y}} \xi(\circ - 2\pi\mathbf{y}) = g + h$$
$$= \sum_{\mathbf{x}\in\mathcal{P}(\mathbf{N})} a_{g,\mathbf{x}} \varphi(\circ - 2\pi\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{P}(\mathbf{N})} a_{h,\mathbf{x}} \psi(\circ - 2\pi\mathbf{x})$$

$\hat{\mathbf{a}}_g, \hat{\mathbf{a}}_h \in \mathbb{C}^{\frac{m}{2}}$ are computed using only $\hat{\mathbf{a}}_f$, $\hat{\mathbf{a}}_\varphi$, and $\hat{\mathbf{a}}_\psi$.

$\Rightarrow$ **Fast wavelet transform (B., 2013)**

Constructing Wavelets

We construct wavelets by their Fourier coefficients. We choose

- a nonnegative function $g : \mathbb{R}^d \to \mathbb{R}_+$ with

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{z}) = 1 \quad \text{and} \quad g(\mathbf{x}) > 0, \quad \text{for all } \mathbf{x} \in [-\tfrac{1}{2}, \tfrac{1}{2})^d$$

- matrices $\mathbf{J}_1, \ldots, \mathbf{J}_n, \mathbf{M}_0 \in \mathbb{Z}^{d\times d}, \quad |\det \mathbf{J}_l| = 2$

And define

- matrices $\mathbf{M}_l := \mathbf{J}_l \ldots \mathbf{J}_1 \mathbf{M}_0$, $m_l := 2^l |\det \mathbf{M}_0|$
- matrix vectors $\mathcal{J}_{l,k} := (\mathbf{J}_l, \ldots, \mathbf{J}_k)$,
- $B_{\mathcal{J}_{l,n}}(\mathbf{x}) := \begin{cases} g(\mathbf{x}) & l = n+1 \\ \Bigl(\sum_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{J}_l^{\mathrm{T}}\mathbf{z})\Bigr) B_{\mathcal{J}_{l+1,n}}(\mathbf{J}_l^{-\mathrm{T}}\mathbf{x}) & l = n, n-1, \ldots, 1 \end{cases}$
- $\tilde{B}_{\mathcal{J}_{l,n}}(\mathbf{x}) := \mathrm{e}^{-2\pi\mathrm{i}\mathbf{x}^{\mathrm{T}}\mathbf{w}_l} \Bigl(\sum_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{J}_l^{\mathrm{T}}\mathbf{z}-\mathbf{v}_l)\Bigr) B_{\mathcal{J}_{l+1,n}}(\mathbf{J}_l^{-\mathrm{T}}\mathbf{x}), \quad l \leq n,$

where $\mathbf{v}_l \in \mathcal{P}(\mathbf{J}_l^{\mathrm{T}})\setminus\{\mathbf{0}\}$ and $\mathbf{w}_l \in \mathcal{P}(\mathbf{J}_l)\setminus\{\mathbf{0}\}$ (both unique)

Constructing Wavelets II

The Multivariate Wavelets of de la Vallée Poussin Type

Definition

Define scaling functions $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}$ and wavelets $\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}$ of de la Vallée Poussin type in Fourier coefficients

$$c_{\mathbf{k}}\left(\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right) := \frac{1}{\sqrt{m_l}} B_{\mathcal{J}_{l+1,n}}(\mathbf{M}_l^{-\mathrm{T}}\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^d, \quad l = 0, \ldots, n$$

$$c_{\mathbf{k}}\left(\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right) := \frac{1}{\sqrt{m_l}} \tilde{B}_{\mathcal{J}_{l+1,n}}(\mathbf{M}_l^{-\mathrm{T}}\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^d, \quad l = 0, \ldots, n-1.$$

Remark

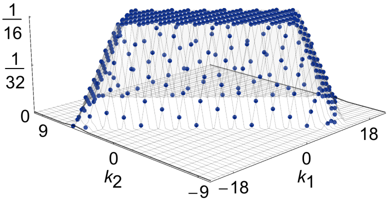
If g is smooth, than $B_{\mathcal{J}_{l,n}}$ is smooth

$c_{\mathbf{k}}\left(\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right)$ and $c_{\mathbf{k}}\left(\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right)$ are samples obtained from a smooth function
$\Rightarrow$ localization

Example of a Wavelet

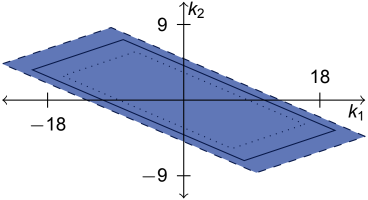
De la Vallée Poussin-type scaling functions and wavelets

- Let g be the Box spline with $\Xi = \begin{pmatrix} 1 & 0 & \frac{1}{10} & 0 \\ 0 & 1 & 0 & \frac{1}{10} \end{pmatrix}$
- $\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix} = \mathbf{J}_X \begin{pmatrix} 14 & -6 \\ 12 & 4 \end{pmatrix}, \quad \mathbf{J}_X = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
- $c_{\mathbf{k}}(\varphi_{\mathbf{M}}) = g(\mathbf{M}^{-\mathrm{T}}\mathbf{k})$, $c_{\mathbf{k}}(\varphi_{\mathbf{N}}^{(\mathbf{J}_X)}) = c_{\mathbf{k}}(\varphi_{\mathbf{N}}) = g(\mathbf{N}^{-\mathrm{T}}\mathbf{k})$



,

$c_{\mathbf{k}}(\varphi_{\mathbf{M}})$



schematic

Example of a Wavelet

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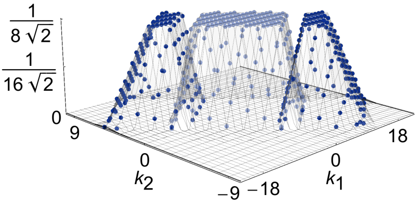
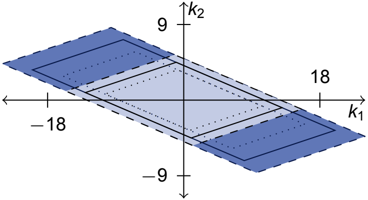


$c_{\mathbf{k}}(\varphi_{\mathbf{N}})$, $|c_{\mathbf{k}}(\psi_{\mathbf{N}})|$



schematic

Example of a Wavelet

De la Vallée Poussin-type scaling functions and wavelets

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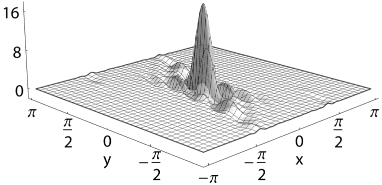

$\varphi_{\mathbf{M}}$

Example of a Wavelet

De la Vallée Poussin-type scaling functions and wavelets

- Let g be the Box spline with $\Xi = \begin{pmatrix} 1 & 0 & \frac{1}{10} & 0 \\ 0 & 1 & 0 & \frac{1}{10} \end{pmatrix}$
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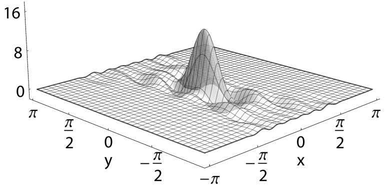
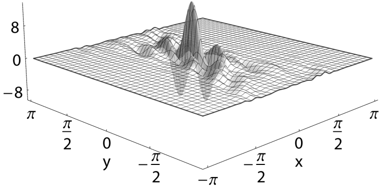


$\varphi_{\mathbf{N}}$



$\psi_{\mathbf{N}}^{(\mathbf{J}_X)}$

Properties of the de la Vallée Poussin Type Wavelets

Theorem (B., J. Prestin, 2014)

For $l = 0, \ldots, n-1$

1. $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} \in \operatorname{span}\left\{\varphi_{\mathbf{M}_{l+1}}^{\mathcal{J}_{l+2,n}}(\circ - 2\pi\mathbf{y})\ ;\ \mathbf{y} \in \mathcal{P}(\mathbf{M}_{l+1})\right\} =: V_{l+1}$
2. $\dim V_{l+1} = |\det \mathbf{M}_{l+1}|$
3. $V_{l+1} = V_l \oplus \operatorname{span}\left\{\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}(\circ - 2\pi\mathbf{y})\ ;\ \mathbf{y} \in \mathcal{P}(\mathbf{M}_l)\right\}.$

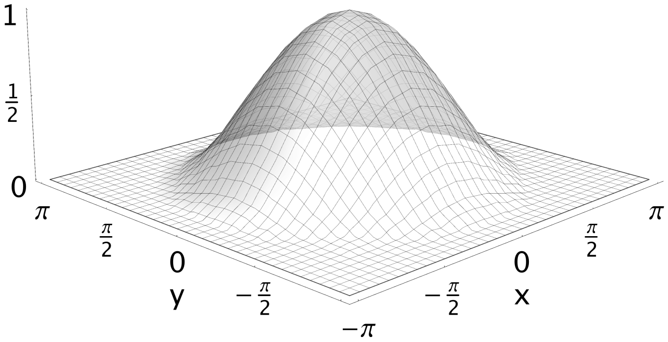
With slight restriction on $g \Rightarrow \psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} = \psi_{\mathbf{M}_l}^{(\mathbf{J}_{l+1})}$ and $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} = \varphi_{\mathbf{M}_l}^{(\mathbf{J}_{l+1})}$.

$\Rightarrow$ With an infinite sequence $\{\mathbf{J}_k\}_{k\in\mathbb{N}}$, $|\det \mathbf{J}_k| = 2$,

the sequence of spaces $\{V_l\}_{l\in\mathbb{N}}$ forms an **MRA**.

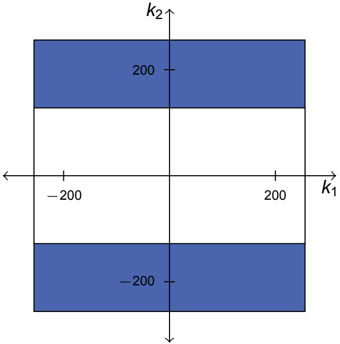
Example of a Decomposition

- radial function based on a piecewise quadratic function
- jump in second directional derivative on a circle
- sampling with $\mathbf{M} = 512\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow 512 \times 512$ pixel image.

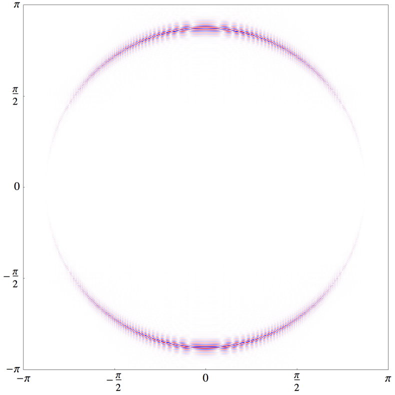


Example of a Decomposition

$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$



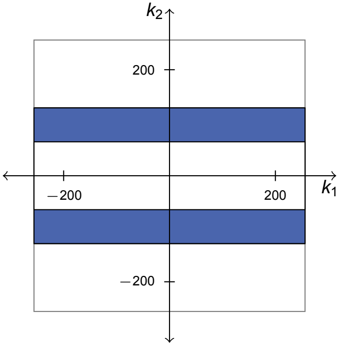
Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_1 = \begin{pmatrix} 512 & 0 \\ 0 & 256 \end{pmatrix}$



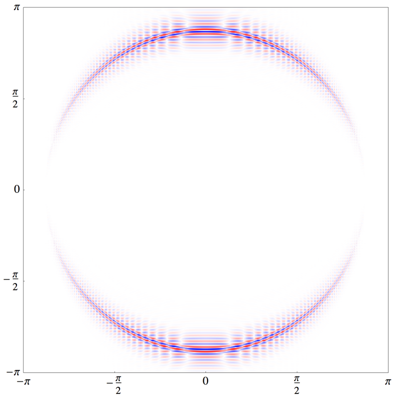
Corresp. wavelet part

Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$



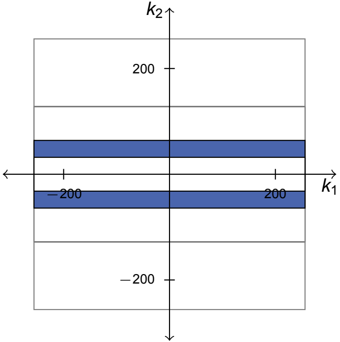
Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_2 = \begin{pmatrix} 512 & 0 \\ 0 & 128 \end{pmatrix}$



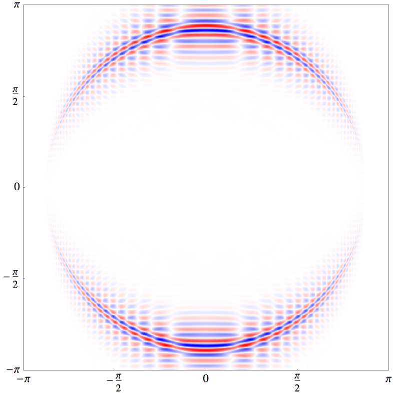
Corresp. wavelet part

Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$



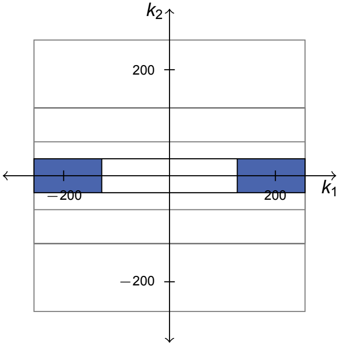
Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_3 = \begin{pmatrix} 512 & 0 \\ 0 & 64 \end{pmatrix}$



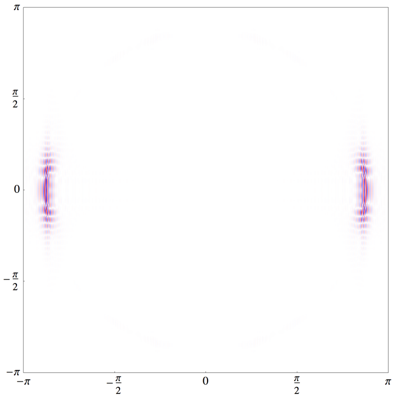
Corresp. wavelet part

Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$



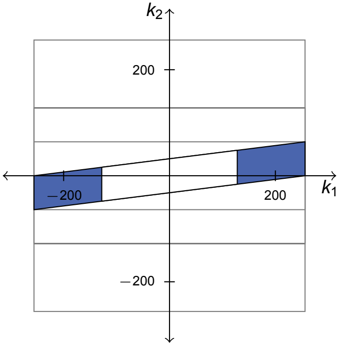
Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_4 = \begin{pmatrix} 256 & 0 \\ 0 & 64 \end{pmatrix}$



Corresp. wavelet part

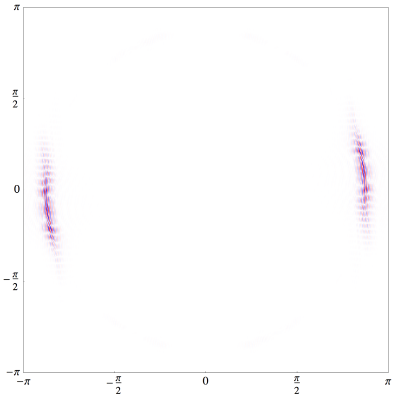
Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$



Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_5 = \begin{pmatrix} 256 & 32 \\ 0 & 64 \end{pmatrix}$

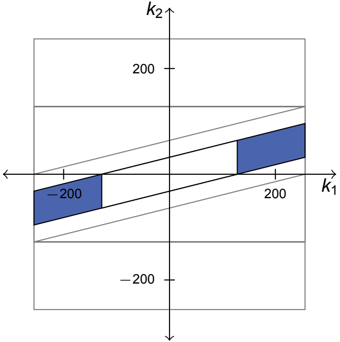
$$\mathbf{J}_{Y-} := \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

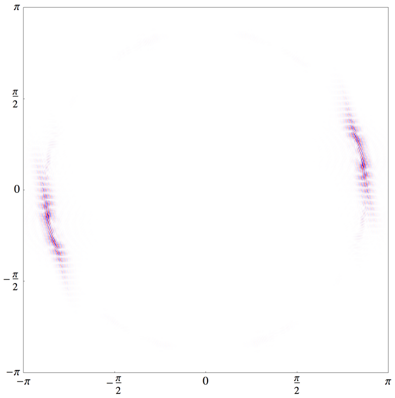


Corresp. wavelet part

Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$





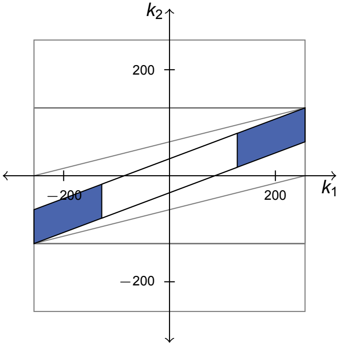
Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_6 = \begin{pmatrix} 256 & 64 \\ 0 & 64 \end{pmatrix}$

$$\mathbf{J}_{Y-} := \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

Corresp. wavelet part

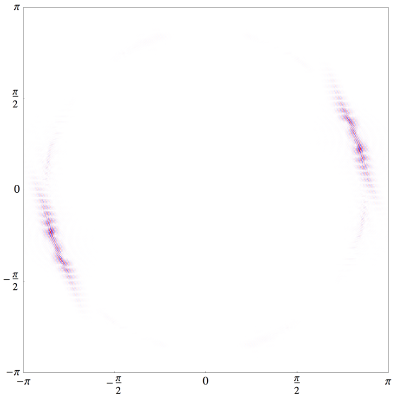
Example of a Decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_{1234567}$$



Wavelet $\psi^{(\mathbf{J}_Y\mathbf{J}_X)}_{\mathbf{N}_{1234567}}$, $\mathbf{N}_7 = \begin{pmatrix} 256 & 96 \\ 0 & 64 \end{pmatrix}$

$$\mathbf{J}_{Y-} := \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$



Corresp. wavelet part

numbering: binary in matrices and directly in multiples of 32 in shear

Conclusion

Patterns $\mathcal{P}(\mathbf{M})$ and generating sets $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$

- generalize equally spaced points
- still resemble an FFT
- fast wavelet transform based with corresponding TI spaces

The constructed wavelets generalize the onedimensional
de la Vallée Poussin wavelets

- to arbitrary dyadic scaling matrices
- based on arbitrary smooth functions g

$\Rightarrow$ localization

- for many functions g we have an MRA

Taking several finite sequences of matrices $\mathbf{J}_l$
$\Rightarrow$ dictionary of directional wavelets

Literature

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Thank your for your attention.