TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

The anisotropic Strang-Fix conditions

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Approximation Methods and Function Spaces
Hasenwinkel

FELIX KLEIN ZENTRUM FÜR MATHEMATIK

*joint work with Jürgen Prestin

Contents

Introduction

Cardinal interpolation on equispaced grids

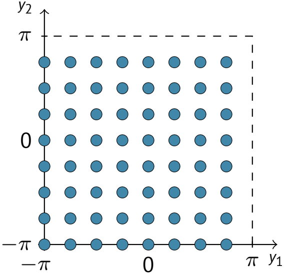
- using polynomials and B-splines on $\mathbb{R}$ [Schönberg, 1969]
- Strang-Fix conditions: quantify reproduction of polynomials [Strang, Fix, 1973]
- tensor product on $\mathbb{R}^d$ & $\mathbb{T}^d$ [Schönberg, 1987; Pöplau, 1995]
- periodic interpolation & Strang-Fix conditions [Pöplau, 1995], [Locher, 1981; Delvos, 1987]
- error of periodic interpolation, e.g. in Besov spaces [Sickel, Sprengel, 1998]

This talk: An anisotropic generalization of

- the norm (spaces) of interest
- the Strang-Fix conditions
- the error of interpolation

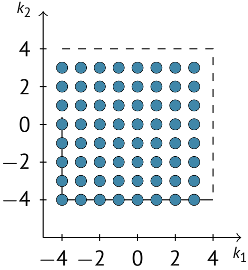
Pattern and the generating Group

Let $N \in \mathbb{N}$ be given



grid

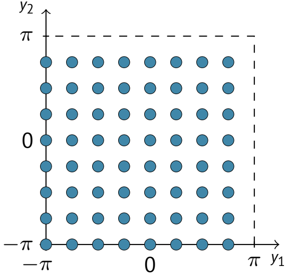
$N = 8$



grid

Pattern and the generating Group

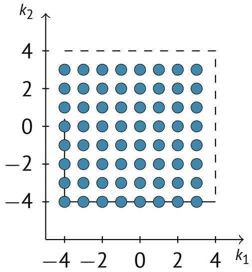
Let $\mathbf{M} = \operatorname{diag}(N,\ldots,N) \in \mathbb{Z}^{d\times d}$ be given



Pattern $2\pi\mathcal{P}(\mathbf{M})$

$$\mathcal{P}(\mathbf{M}) := \left[-\tfrac{1}{2},\tfrac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$$

$$\mathbf{M} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

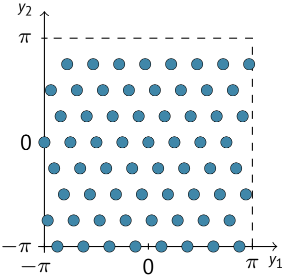


generating Set

$$\mathcal{G}(\mathbf{M}^{\mathrm{T}}) := \mathbf{M}^{\mathrm{T}}\mathcal{P}(\mathbf{M}^{\mathrm{T}}) = \mathbf{M}^{\mathrm{T}}\left[-\tfrac{1}{2},\tfrac{1}{2}\right)^d \cap \mathbb{Z}^d$$

Pattern and the generating Group

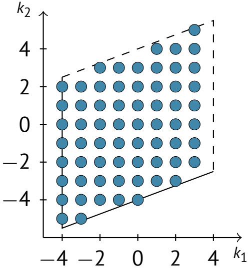
Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ regular be given



Pattern $2\pi\mathcal{P}(\mathbf{M})$

$$\mathcal{P}(\mathbf{M}) := \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$$

$$\mathbf{M} = \begin{pmatrix} 8 & 3 \\ 0 & 8 \end{pmatrix}$$



generating Set

$$\mathcal{G}(\mathbf{M}^{\mathrm{T}}) := \mathbf{M}^{\mathrm{T}}\mathcal{P}(\mathbf{M}^{\mathrm{T}}) = \mathbf{M}^{\mathrm{T}}\left[-\tfrac{1}{2}, \tfrac{1}{2}\right)^d \cap \mathbb{Z}^d$$

- $m := |\mathcal{P}(\mathbf{M})| = |\mathcal{G}(\mathbf{M})| = |\det \mathbf{M}|$
- $\big(\mathcal{P}(\mathbf{M}), + \bmod 1\big)$ is a group.

Fourier Transform and Partial Sum on $\mathcal{P}(\mathbf{M})$

For $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^m$ (fixed ordering): DFT

$$\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} := \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a} \in \mathbb{C}^m,$$

where the Fourier matrix (same ordering of columns) is given by

$$\mathcal{F}(\mathbf{M}) := \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}}),\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^{m\times m}$$

Fourier Transform and Partial Sum on $\mathcal{P}(\mathbf{M})$

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Using the Fourier coefficients of $f \in L_1(\mathbb{T}^d)$ defined as usual

$$c_{\mathbf{k}}(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d,$$

we define the Fourier partial sum $\mathrm{S}_{\mathbf{M}} f := \sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} c_{\mathbf{h}}(f)\mathrm{e}^{\mathrm{i}\mathbf{h}^{\mathrm{T}}\circ} \in \mathcal{T}_{\mathbf{M}},$

which is a trigonometric polynomial on $\mathcal{G}(\mathbf{M})$, i.e.

$$\mathcal{T}_{\mathbf{M}} := \left\{f;\, f = \sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \hat{a}_{\mathbf{h}}\mathrm{e}^{\mathrm{i}\mathbf{h}^{\mathrm{T}}\circ}, \quad \hat{a}_{\mathbf{h}} \in \mathbb{C}\right\}.$$

Translation Invariant Spaces

For $\varphi \in L_1(\mathbb{T}^d)$ we define the translation invariant (TI) space w.r.t. $\mathcal{P}(\mathbf{M})$:

$$V_{\mathbf{M}}^{\varphi} := \left\{ f;\, f = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{f,\mathbf{y}}\varphi(\circ - 2\pi\mathbf{y}), \quad \mathbf{a}_f = (a_{f,\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^m \right\}$$

This can be expressed in Fourier coefficients

Lemma

$f \in V_{\mathbf{M}}^{\varphi}$ if and only if

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(f) = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{f,\mathbf{y}} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\varphi) = \hat{a}_{f,\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\varphi),$$

holds for all $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$, $\mathbf{z} \in \mathbb{Z}^d$, where $\hat{\mathbf{a}}_f = \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a}_f$.

Function Spaces

For $\beta \geq 0$, $q \geq 1$ define the spaces [Sprengel, 1998]

$$A_q^\beta(\mathbb{T}^d) := \left\{ f \in \mathrm{L}_1(\mathbb{T}^d) \,\middle|\, \left\| f \,\middle|\, A_{,q}^\beta \right\| < \infty \right\},$$

where

$$\left\| f \,\middle|\, A_q^\beta \right\| := \left\| \{ (1 + \|\mathbf{k}\|_2^2)^{\beta/2} c_{\mathbf{k}}(f) \}_{\mathbf{k} \in \mathbb{Z}^d} \,\middle|\, \ell_q(\mathbb{Z}^d) \right\|.$$

- $q = 2$: Sobolev spaces $H^\beta(\mathbb{T}^d) = A_2^\beta(\mathbb{T}^d)$
- smoothness imposed by *isotropic decay* of Fourier coefficients $c_{\mathbf{k}}(f)$
- Wiener Algebra $A_1^0(\mathbb{T}^d)$

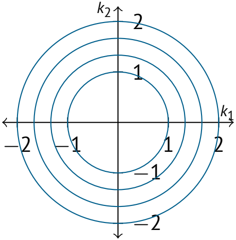
Function Spaces

For $\beta \geq 0$, $q \geq 1$ define the spaces [B., Prestin, 2014]

$$A^{\beta}_{\mathbf{E}_d,\,q}(\mathbb{T}^d) := \left\{ f \in \mathrm{L}_1(\mathbb{T}^d) \,\middle|\, \left\| f \,\middle|\, A^{\beta}_{\mathbf{E}_d,q} \right\| < \infty \right\},$$

where

$$\left\| f \,\middle|\, A^{\beta}_{\mathbf{E}_d,q} \right\| := \left\| \{\sigma^{\mathbf{E}_d}_{\beta}(\mathbf{k}) c_{\mathbf{k}}(f)\}_{\mathbf{k}\in\mathbb{Z}^d} \,\middle|\, \ell_q(\mathbb{Z}^d) \right\|.$$



niveau lines of $\sigma^{N\mathbf{E}_d}_{\beta}(\mathbf{k}) = (1 + \|N\mathbf{E}_d\|_2^2 \|\frac{1}{N}\mathbf{k}\|_2^2)^{\beta/2}$

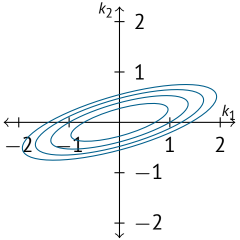
Function Spaces

For $\beta \geq 0$, $q \geq 1$ define the spaces [B., Prestin, 2014]

$$A^{\beta}_{\mathbf{M},q}(\mathbb{T}^d) := \left\{ f \in \mathrm{L}_1(\mathbb{T}^d) \,\middle|\, \big\| f \,\big|\, A^{\beta}_{\mathbf{M},q} \big\| < \infty \right\},$$

where

$$\big\| f \,\big|\, A^{\beta}_{\mathbf{M},q} \big\| := \Big\| \{\sigma^{\mathbf{M}}_{\beta}(\mathbf{k}) c_{\mathbf{k}}(f)\}_{\mathbf{k}\in\mathbb{Z}^d} \,\Big|\, \ell_q(\mathbb{Z}^d) \Big\|.$$



niveau lines of $\sigma^{\mathbf{M}}_{\beta}(\mathbf{k}) := \left(1 + \|\mathbf{M}\|_2^2 \|\mathbf{M}^{-\mathrm{T}}\mathbf{k}\|_2^2\right)^{\beta/2}$, $\quad \mathbf{k} \in \mathbb{Z}^d, \mathbf{M} = \begin{pmatrix} 16 & 0 \\ 14 & 8 \end{pmatrix}$

Anisotropic decay, but equivalent norms for fixed q, β, i.e. $A^{\beta}_{q} = A^{\beta}_{\mathbf{M},q}$.

Interpolation

- Sample a function: $a_{\mathbf{y}} = f(2\pi\mathbf{y}),\ \mathbf{y} \in \mathcal{P}(\mathbf{M})$
- Interpolation operator: $\mathrm{L}_{\mathbf{M}} f \in V_{\mathbf{M}}^{\varphi}$, i.e. $\mathrm{L}_{\mathbf{M}} f(2\pi\mathbf{y}) = f(2\pi\mathbf{y})$
- for cardinal interpolant $\mathrm{I}_{\mathbf{M}} \in V_{\mathbf{M}}^{\varphi}$:

$$\mathrm{L}_{\mathbf{M}} f = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} f(2\pi\mathbf{y})\, \mathrm{I}_{\mathbf{M}}(\circ - 2\pi\mathbf{y}).$$

Lemma

Let $\varphi \in A(\mathbb{T}^d)$ and $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be regular. Then $\mathrm{I}_{\mathbf{M}} \in V_{\mathbf{M}}^{\varphi}$ exists iff

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\varphi) \neq 0, \quad \text{for all } \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}).$$

Sketch of proof: Use $c_{\mathbf{k}}(\mathrm{I}_{\mathbf{M}})$, discrete Fourier coefficients & Aliasing formula:

$$c_{\mathbf{k}}^{\mathbf{M}}(\varphi) := \frac{1}{m} \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \varphi(2\pi\mathbf{y})\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{y}} = \sum_{\mathbf{z}\in\mathbb{Z}^d} c_{\mathbf{k}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\varphi), \quad \mathbf{k} \in \mathbb{Z}^d$$

Periodic Strang-Fix Conditions

Definition (Periodic Strang-Fix Conditions)

For $N \in \mathbb{N}$, $s > 0$, $q \geq 1$ and an $\alpha \in \mathbb{R}^+$,
the cardinal interpolant $\mathrm{I}_N \in \mathrm{L}_1(\mathbb{T}^d)$ fulfills the
periodic Strang-Fix conditions of order s,
if there exists a nonnegative sequence $\mathbf{b} = \{b_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$, s.t.

1. $|1 - N^d c_{\mathbf{h}}(\mathrm{I}_N)| \leq b_{\mathbf{0}} N^{-s} \|\mathbf{h}\|_2^s$,
2. $|N^d c_{\mathbf{h}+N\mathbf{z}}(\mathrm{I}_N)| \leq b_{\mathbf{z}} N^{-s-\alpha} \|\mathbf{h}\|_2^s$

holds for any $\mathbf{h} \in [-\frac{N}{2}, \ldots, \frac{N}{2} - 1]^d$, $\mathbf{z} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$, and

$$\gamma_{\mathrm{SF}} := \|\{\sigma_\alpha^{\mathbf{E}_d}(\mathbf{z}) b_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d} | \ell_q(\mathbb{Z}^d)\| < \infty.$$

Anisotropic Periodic Strang-Fix Conditions

Definition (Anisotropic Periodic Strang-Fix Conditions)

For $\mathbf{M} \in \mathbb{Z}^{d\times d}$, $\lambda_1(\mathbf{M}) > 1$, $s > 0$, $q \geq 1$ and an $\alpha \in \mathbb{R}^+$, the cardinal interpolant $\mathrm{I}_{\mathbf{M}} \in \mathrm{L}_1(\mathbb{T}^d)$ fulfills the *elliptic/anisotropic periodic Strang-Fix conditions of order s*, if there exists a nonnegative sequence $\mathbf{b} = \{b_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$, s.t.

1. $|1 - mc_{\mathbf{h}}(\mathrm{I}_{\mathbf{M}})| \leq b_{\mathbf{0}}\kappa_{\mathbf{M}}^{-s}\|\mathbf{M}^{-\mathrm{T}}\mathbf{h}\|_2^s$,
2. $|mc_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\mathrm{I}_{\mathbf{M}})| \leq b_{\mathbf{z}}\kappa_{\mathbf{M}}^{-s}\|\mathbf{M}\|_2^{-\alpha}\|\mathbf{M}^{-\mathrm{T}}\mathbf{h}\|_2^s$

holds for any $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$, $\mathbf{z} \in \mathbb{Z}^d\backslash\{\mathbf{0}\}$, and

$$\gamma_{\mathrm{SF}} := \|\{\sigma_\alpha^{\mathbf{M}}(\mathbf{z})b_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}|\ell_q(\mathbb{Z}^d)\| < \infty.$$

$\kappa_{\mathbf{M}} := \|\mathbf{M}\|_2\|\mathbf{M}^{-1}\|_2$ denotes the condition number of $\mathbf{M}$.

Error Bounds for Interpolation

First Step: Triangle Inequality

- let f have a certain anisotropic smoothness, i.e. $f \in A^{\alpha}_{\mathbf{M},q}$
- let $\mathrm{I}_{\mathbf{M}} \in V^{\varphi}_{\mathbf{M}}$ fulfill the anisotropic Strang-Fix conditions of order $s \geq 0$ for $q, \alpha, \mathbf{M}$
- Goal: Upper bound for $\|f - \mathrm{L}_{\mathbf{M}}f\|$ in certain norm $\|\cdot\|$

Idea: If f is very smooth along a certain direction, then a few translates should be sufficient, and vice versa many translates for “rough directions”.

Instead of the pattern $\mathcal{P}(\mathbf{M})$: Take a “good” set of trig. polynomials $\mathcal{T}_{\mathbf{M}}$.

First step for the upper bound of $\|f - \mathrm{L}_{\mathbf{M}}f\|$: triangle inequality

$$\|f - \mathrm{L}_{\mathbf{M}}f\| \leq \|\mathrm{S}_{\mathbf{M}}\, f - \mathrm{L}_{\mathbf{M}}\, \mathrm{S}_{\mathbf{M}}\, f\| + \|f - \mathrm{S}_{\mathbf{M}}\, f\| + \|\mathrm{L}_{\mathbf{M}}(f - \mathrm{S}_{\mathbf{M}}\, f)\|$$

Error Bound for Interpolation

Part I: trigonometric polynomials

Theorem (B., Prestin, 2014)

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$, $\lambda_1(\mathbf{M}) > 1$, $g \in \mathcal{T}_{\mathbf{M}}$ *and let* $\mathrm{I}_{\mathbf{M}} \in A(\mathbb{T}^d)$ *corresp. to* φ *fulfill the Strang-Fix cond. for* $s \geq 0$, $\alpha > 0$ *and* $q \geq 1$. *Then*

$$\left\|g - \mathrm{L}_{\mathbf{M}} g \,\middle|\, A^{\alpha}_{\mathbf{M},q}\right\| \leq \left(\frac{1}{\|\mathbf{M}\|_2}\right)^{s} \gamma_{\mathrm{SF}} \left\|g \,\middle|\, A^{\alpha+s}_{\mathbf{M},q}\right\|.$$

- proof: take $c_{\mathbf{k}}(g - \mathrm{L}_{\mathbf{M}} g)$ and apply Strang-Fix condition inequalities
- apply to $g = \mathrm{S}_{\mathbf{M}} f$ and use $\left\|\mathrm{S}_{\mathbf{M}} f \,\middle|\, A^{\alpha+s}_{\mathbf{M},q}\right\| \leq \left\|f \,\middle|\, A^{\alpha+s}_{\mathbf{M},q}\right\|$.
- $\|\mathbf{M}\|_2$ denotes length of main axis of the ellipsoid $\|\mathbf{M}^{-\mathrm{T}}\mathbf{x}\|_2 = 1$
- if f is smooth along this direction, the left hand side is very small.

Error Bound for Interpolation

Part II & III: Error of approximation with Fourier partial sum and its interpolant

Theorem (B., Prestin, 2014)

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ *regular,* $f \in A^{\mu}_{\mathbf{M},q}(\mathbb{T}^d)$, $q \geq 1$ *und* $\mu \geq \alpha \geq 0$. *Then*

$$\left\| f - S_{\mathbf{M}} f \,\middle|\, A^{\alpha}_{\mathbf{M},q} \right\| \leq \left(\frac{2}{\|\mathbf{M}\|_2} \right)^{\mu-\alpha} \left\| f \,\middle|\, A^{\mu}_{\mathbf{M},q} \right\|.$$

Sketch of proof: Split weight $\sigma^{\mathbf{M}}_{\alpha}(\mathbf{k}) = \sigma^{\mathbf{M}}_{\alpha-\mu}(\mathbf{k})\sigma^{\mathbf{M}}_{\mu}(\mathbf{k})$
and bound first term from above for all $\mathbf{k} \in \mathbb{Z}^d \backslash \mathcal{G}(\mathbf{M}^{\mathrm{T}})$.

Error Bound for Interpolation

Part II & III: Error of approximation with Fourier partial sum and its interpolant

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Sketch of proof: Split weight $\sigma^{\mathbf{M}}_{\alpha}(\mathbf{k}) = \sigma^{\mathbf{M}}_{\alpha-\mu}(\mathbf{k})\sigma^{\mathbf{M}}_{\mu}(\mathbf{k})$ and bound first term from above for all $\mathbf{k} \in \mathbb{Z}^d \backslash \mathcal{G}(\mathbf{M}^{\mathrm{T}})$.

Theorem (B., Prestin, 2014)

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ *be regular,* $f \in A^{\mu}_{\mathbf{M},q}(\mathbb{T}^d)$, $q \geq 1$, $\mu \geq \alpha \geq 0$, *and* $\mu > d(1 - 1/q)$. *Then*

$$\left\| \mathrm{L}_{\mathbf{M}}(f - \mathrm{S}_{\mathbf{M}} f) \,\middle|\, A^{\alpha}_{\mathbf{M},q} \right\| \leq \gamma_{\mathrm{IP}}\gamma_{\mathrm{Sm}} \left(\frac{1}{\|\mathbf{M}\|_2} \right)^{\mu-\alpha} \left\| f \,\middle|\, A^{\mu}_{\mathbf{M},q} \right\|,$$

where γ_{IP} *does only depends on* $\mathrm{I}_{\mathbf{M}}$, *i.e. on* φ, *and* γ_{Sm} *only on* q, α *and* μ.

Error Bound for Interpolation

Putting the three parts together

Theorem (B., Prestin, 2014)

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$, $\lambda_1(\mathbf{M}) > 1$ *and* $f \in A^{\mu}_{\mathbf{M},q}(\mathbb{T}^d)$, $\mu \geq \alpha \geq 0$, *with* $\mu > d(1-1/q)$. *Let the cardinal interpolant* $\mathrm{I}_{\mathbf{M}}$ *corresp. to* φ *fulfill the anisotropic Strang Fix conditions of order* $s > 0$, *and* $q \geq 1$, $\alpha \geq 0$. *Then*

$$\left\| f - \mathrm{L}_{\mathbf{M}} f \,\middle|\, A^{\alpha}_{\mathbf{M},q} \right\| \leq C_\rho \left(\frac{1}{\|\mathbf{M}\|_2} \right)^{\rho} \left\| f \,\middle|\, A^{\mu}_{\mathbf{M},q} \right\|, \quad \text{where } \rho := \min\{s, \mu - \alpha\},$$

$$C_\rho := \begin{cases} \gamma_{\mathrm{SF}} + 2^{\mu-\alpha} + \gamma_{\mathrm{IP}}\gamma_{\mathrm{Sm}} & \text{for } \rho = s, \\ (1+d)^{s+\alpha-\mu}\gamma_{\mathrm{SF}} + 2^{\mu-\alpha} + \gamma_{\mathrm{IP}}\gamma_{\mathrm{Sm}} & \text{for } \rho = \mu - \alpha. \end{cases}$$

- For $\rho = \mu - \alpha$: similar theorem to part I necessary.
- $\mu - \alpha$ “additional smoothness” of f compared to error of interpolation
- the Strang-Fix order s of φ: saturation level

Conclusion & Future Work

- added anisotropy to the grid $\Rightarrow$ pattern/generating set
- adapted (anisotropic) periodic Strang-Fix conditions
- classification/introduction of directions
  - major axes of ellipsoids $\|\mathbf{M}^{-\mathrm{T}}\mathbf{x}\| = c$ in Fourier coefficient indices
  - $\mathbf{M}^{-1}\mathbf{v}_j$ in time domain
- upper bound for error of interpolation of $f \in A^{\alpha}_{\mathbf{M},q}$.

Future Work

- Are the anisotropic sparse grids?
- extend approach to anisotropic spaces of mixed smoothness
- application to static linear elasticity on a periodic composite

Literature

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[2] RB and J. Prestin. Multivariate Periodic Wavelets of de la Vallée Poussin Type. *Journal of Fourier Analysis and Applications*, 2015.

[3] RB. Translationsinvariante Räume multivariater anisotroper Funktionen auf dem Torus. *Dissertation*, 2013.

[4] G. Pöplau. Multivariate periodische Interpolation durch Translate und deren Anwendung. *Dissertation*, 1995.

[5] W. Sickel and F. Sprengel. Some error estimates for periodic interpolation of functions from Besov spaces. *Advances in multivariate approximation 1998*, 1999.

[6] F. Sprengel. Interpolation und Waveletzerlegung multivariater periodischer Funktionen. *Dissertation*, 1997.

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Thank you for your attention.