

# A Second Order Non-smooth Variational Model for Restoring Manifold-valued Images

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\* joint work with M. Bačák (MPI MIS Leipzig),  
G. Steidl (U Kaiserslautern), A. Weinmann (TU Munich)



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- 2 Second Order Differences on Manifolds
- 3 Proximal Mappings and the Cyclic Proximal Point Algorithm
- 4 Examples
- 5 Conclusion

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# Restoring Images: Denoising and Inpainting

Introducing discrete total variation models for real valued data.

- given noisy image  $f: \mathcal{V} \rightarrow \mathbb{R}$ ,  $\mathcal{V} \subseteq \mathcal{G} = \{1, \dots, N\} \times \{1, \dots, M\}$
- reconstruct original image  $u_0: \mathcal{G} \rightarrow \mathbb{R}$
- pixel from  $\mathcal{G} \setminus \mathcal{V}$  have to be inpainted
- approach: Minimize variational model

$$\mathcal{E}(u) := \underbrace{F(f, u)}_{\text{data term}} + \underbrace{\alpha R(u)}_{\text{regularization term}}, \quad \alpha > 0$$

- first order models (total variation, TV)

[Rudin, Osher, Fatemi, 1992]

- isotropic model with discrete gradient  $|\nabla u|$ :  $R_{\text{iso}}(u) := \sum_{i,j} |\nabla u|$
- anisotropic model:  $R_{\text{aniso}}(u) := \sum_{i,j} (|u_{i+1,j} - u_{i,j}| + |u_{i,j+1} - u_{i,j}|)$
- edge preserving

- several higher order variational models avoid **stair causing-effect**

[Chambolle, Lions, 1997; Setzer, Steidl, 2008; Bredies, Kunisch, Pock, 2010; Papafitsoros, Schönlieb, 2014]

# Riemannian Manifolds

...which we want to apply in different spaces

Here: Images with pixel values in a Riemannian manifold  $\mathcal{M}$

Goal: Second Order Model for images  $f: \mathcal{V} \rightarrow \mathcal{M}$

Applications with manifold-valued images are for example

$\mathbb{S}^1$  InSAR, HSI(HSV) color space, phase space

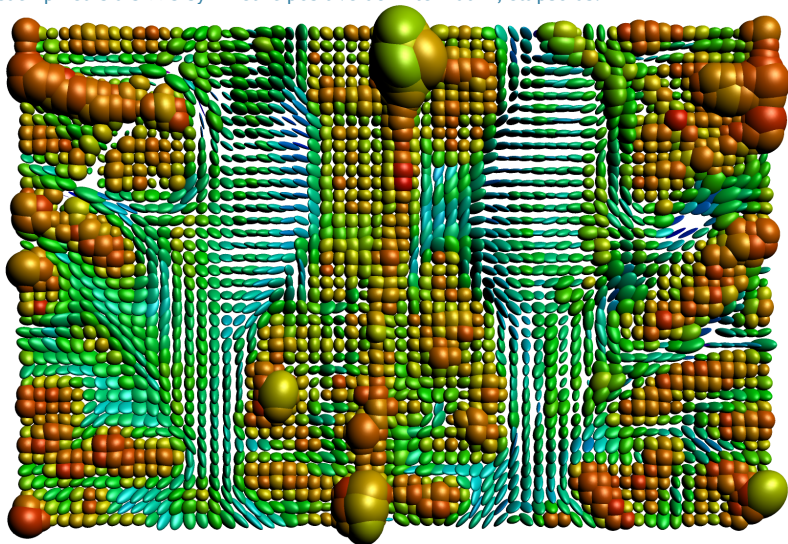
$\mathbb{S}^2$  directions, chromaticity-brightness colorspace

$SO(3)$  orientations, electron backscattered diffraction

$\mathcal{P}(s)$  DT-MRI, covariance matrices

# Symmetric Positive Definite Matrices: $x^T Ax > 0$

$\mathcal{P}(3)$ : each pixel is a  $3 \times 3$  symmetric positive definite matrix, ellipsoids:  $x^T Ax = 1$



1 Introduction

**2 Second Order Differences on Manifolds**

3 Proximal Mappings and the Cyclic Proximal Point Algorithm

4 Examples

5 Conclusion

# First and Second Order Differences

Let's restrict ourselves to signals for now...

$$\mathcal{G} = \{1, \dots, N\}, \mathcal{V} \subset \mathcal{G}$$

On  $\mathbb{R}^n$

- line  $\gamma(t) = x + t(y - x)$
- distance  $\|x - y\|_2$
- first order model

$$\sum_{i \in \mathcal{V}} \|f_i - u_i\|_2^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} \|u_i - u_{i+1}\|_2$$

Riemannian manifold  $\mathcal{M}$

- geodesic path  $\gamma_{\widehat{x,y}}(t)$
- geodesic distance  $d: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$
- first order model

[Lellmann et al., 2013; Weinmann et. al., 2014]

$$\sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1})$$



# First and Second Order Differences

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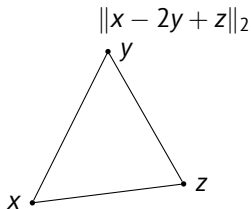
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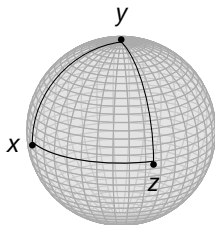
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$$\sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1})$$

- How to model that on  $\mathcal{M}$ ?



$$\mathcal{M} = \mathbb{S}^2$$

# First and Second Order Differences

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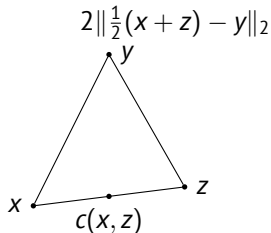
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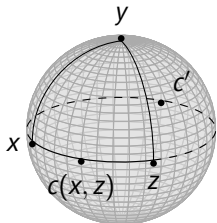
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- **idea:** mid point formulation



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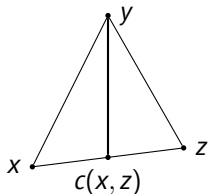
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- second order difference

$2\|c(x, z) - y\|_2$ ,  $c(x, y)$ : mid point



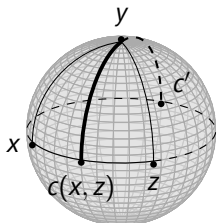
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$$\mathcal{M} = \mathbb{S}^2$$

# A Second Order TV-type Model

Mid points between  $x, z \in \mathcal{M}$ :

$$\mathcal{C}_{x,z} := \left\{ c \in \mathcal{M} : c = \gamma_{\widehat{x,z}}\left(\frac{T}{2}\right) \text{ for any geodesic } \gamma_{\widehat{x,z}}, T := \mathcal{L}(\gamma_{\widehat{x,z}}) \right\}$$

The **Absolute Second Order Difference**:

$$d_2(x, y, z) := \min_{c \in \mathcal{C}_{x,z}} d(c, y), \quad x, y, z \in \mathcal{M}.$$

⇒ **Second Order TV-type Model** for  $\mathcal{M}$ -valued signals  $f$

$$\mathcal{E}(u) := \sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1}) + \beta \sum_{i \in \mathcal{G} \setminus \{1, N\}} d_2(u_{i-1}, u_i, u_{i+1})$$

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# Proximal Mappings and the CPPA

How to minimize the Second Order TV-type Model on  $\mathcal{M}$ ?

To compute a minimizer of  $\mathcal{E}(u)$ : **proximal mappings**

For a function  $\varphi: \mathcal{M}^n \rightarrow (-\infty, +\infty]$  and  $\lambda > 0$  the **proximal mapping** is defined by

[Moreau, 1965; Rockafellar, 1976; Ferreira, Oliveira, 2002]

$$\text{prox}_{\lambda\varphi}(g) := \arg \min_{u \in \mathcal{M}^n} \frac{1}{2} \sum_{i=1}^n d(u_i, g_i)^2 + \lambda\varphi(u).$$

**Note:** For a minimizer  $u^*$  of  $\varphi$  it holds  $\text{prox}_{\lambda\varphi}(u^*) = u^*$ .

For  $\varphi = \sum_{l=1}^c \varphi_l$  use **Cyclic Proximal Point Algorithm (CPPA)** [Bertsekas, 2011; Bačák, 2014]

$$x^{(k+\frac{l+1}{c})} = \text{prox}_{\lambda_k \varphi_l}(x^{(k+\frac{l}{c})}), \quad i = 0, \dots, c-1, k > 0$$

For **real-valued data**: Converges to a minimizer if  $\{\lambda_k\}_{k \in \mathbb{N}} \in \ell_2(\mathbb{Z}) \setminus \ell_1(\mathbb{Z})$ .

# Minimizing the Second Order Model

Applying the CPPA to our model on manifolds.

To minimize

$$\mathcal{E}(u) := \sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1}) + \beta \sum_{i \in \mathcal{G} \setminus \{1, N\}} d_2(u_{i-1}, u_i, u_{i+1})$$

take each summand as one  $\varphi_l$  in the CPPA.

For the involved proximal maps we have

- $\varphi(u_i) = d(f_i, u_i)^2$ : analytical solution [Ferreira, Oliveira, 2002]
- $\varphi(u_i, u_{i+1}) = d(u_i, u_{i+1})$ : analytical solution [Weinmann, Storath, Demaret, 2014]
- $\varphi(u_{i-1}, u_i, u_{i+1}) = d_2(u_{i-1}, u_i, u_{i+1})$ :
  - analytical solution for  $\mathbb{S}^1$  [RB, Laus, Steidl, Weinmann, 2014]
  - numerical solution otherwise [Bačák, RB, Steidl, Weinmann, 2016]

# Proximal Mapping of a Second Order Difference on $\mathcal{M}$

To compute

$$\text{prox}_{\lambda d_2}(g) = \arg \min_{u \in \mathcal{M}^3} \left\{ \frac{1}{2} \sum_{i=1}^3 d(u_i, g_i)^2 + \lambda d_2(u_1, u_2, u_3) \right\}$$

(sub)gradient descent method requires gradient of  $d_2(x, y, z) = d(c(x, z), y)$

$$\nabla_{\mathcal{M}^3} d_2 = (\nabla_{\mathcal{M}} d_2(\cdot, y, z), \nabla_{\mathcal{M}} d_2(x, \cdot, z), \nabla_{\mathcal{M}} d_2(x, y, \cdot))^T.$$

For  $y \neq c(x, z)$  and an ONB  $\{\xi_1, \dots, \xi_n\}$  of  $T_x \mathcal{M}$

- $\nabla_{\mathcal{M}} d_2(x, \cdot, z)(y) = \frac{\log_y c(x, z)}{\|\log_y c(x, z)\|_y} \in T_y \mathcal{M}$
- $\nabla_{\mathcal{M}} d_2(\cdot, y, z)$  by chain rule

$$\nabla_{\mathcal{M}} d_2(\cdot, y, z)(x) = \sum_{k=1}^n \left\langle \frac{\log_{c(x)} y}{\|\log_{c(x)} y\|_{c(x)}}, D_x c[\xi_k] \right\rangle_{c(x)}, \quad \xi_k \in T_x \mathcal{M}.$$



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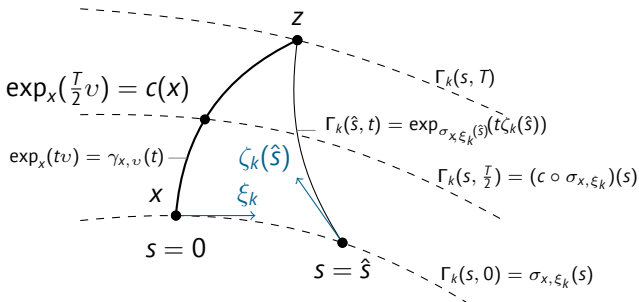
$$\nabla_{\mathcal{M}} d_2(\cdot, y, z)(x) = \sum_{k=1}^n \left\langle \frac{\log_{c(x)} y}{\|\log_{c(x)} y\|_{c(x)}}, D_x c[\xi_k] \right\rangle_{c(x)}, \quad \xi_k \in T_x \mathcal{M}.$$

# Geodesic Variation

How do small changes in  $x$  affect the geodesic?

- $\sigma_{x, \xi_k}(s)$ ,  $k = 1, \dots, n$ ,  $s \in (-\varepsilon, \varepsilon)$ :  
curve starting in  $\sigma_{x, \xi_k}(0) = x$  with direction  $\sigma'_{x, \xi_k}(0) = \xi_k$
  - $\zeta_k(s)$ : direction in  $\sigma_{x, \xi_k}(s)$  towards  $z$ .
- ⇒ geodesic variation of the geodesic  $\gamma_{x, z}$

$$\Gamma_k(s, t) := \exp_{\sigma_{x, \xi_k}(s)}(t\zeta_k(s)), \quad s \in (-\varepsilon, \varepsilon), \quad t \in [0, T], \quad T = d(x, z)$$



# Jacobi Fields

Indicate directions of change along the geodesic and rephrase computation of the gradient to...

The Jacobi field  $J_k$ ,  $k = 1, \dots, n$ , along  $\gamma_{\widehat{x}, \widehat{z}}$  is defined by

$$J_k(t) := \frac{\partial}{\partial s} \Gamma_k(s, t) \Big|_{s=0}, \quad t \in [0, T]$$

## Lemma

$$D_x c[\xi_k] = J_k\left(\frac{T}{2}\right)$$

## Lemma

A Jacobi field  $J_k$  fulfills the System of ODEs

$$\frac{D^2}{dt^2} J_k + R(J_k, \dot{\gamma}_{\widehat{x}, \widehat{z}}) \dot{\gamma}_{\widehat{x}, \widehat{z}} = 0, \quad J_k(0) = \xi_k, \quad J_k(T) = 0$$

with the Riemannian curvature tensor  $R: TM \times TM \times TM \rightarrow TM$ .

# Riemannian Symmetric Spaces

At least in beautiful Spaces we can solve the system of ODEs

- For every  $x \in \mathcal{M}$  exists an isometry  $s_x$  on  $\mathcal{M}$ , i.e.

$$s_x(x) = x, D_x s[\xi] = -\xi, \quad \text{for all } \xi \in T_x \mathcal{M}$$

- covariant derivative

$$\nabla R = 0$$

⇒ the system of ODEs simplifies to a **linear system of ODEs with constant coefficients**

- Examples of symmetric spaces
  - Spheres  $\mathbb{S}^n$
  - Hyperbolic Spaces
  - positive definite matrices  $\mathcal{P}(s)$
  - Grasmannians

*Riemannian symmetric spaces are the most beautiful and most important Riemannian manifolds.*

## The Derivative $D_x c[\xi_k]$ in Symmetric Spaces

Finally we are able to evaluate the Jacobi field at the mid point  $T/2$

Given a specific ONB  $\{\xi_1, \dots, \xi_n\}$  at  $t = 0$  and its parallel transport  $\{\Theta_1(t), \dots, \Theta_n(t)\}$  along the geodesic, i.e.

$$J_k(t) = \sum_{i=1}^n a_{k,i}(t) \Theta_i(t).$$

### Corollary

With  $T = d(x, z)$  and  $\kappa_i$  “characterize curvature” we have for  $k = 1, \dots, n$

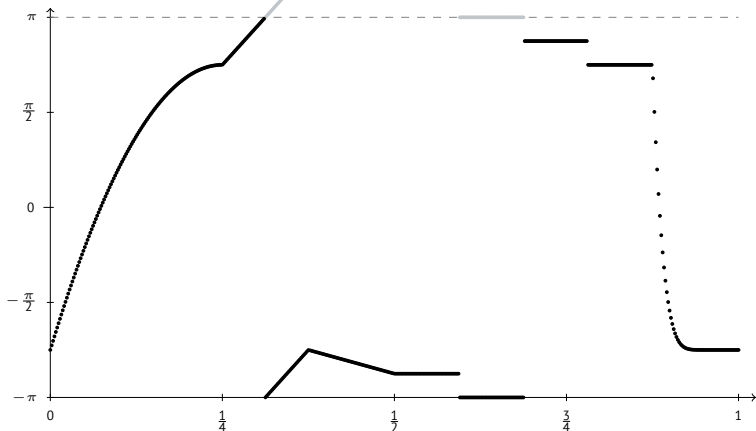
$$D_x c[\xi_k] = J_k\left(\frac{T}{2}\right) = \begin{cases} \frac{\sinh\left(\sqrt{-\kappa_k} \frac{T}{2}\right)}{\sinh(\sqrt{-\kappa_k} T)} \Theta_k\left(\frac{T}{2}\right), & \text{if } \kappa_k < 0, \\ \frac{\sin\left(\sqrt{\kappa_k} \frac{T}{2}\right)}{\sin(\sqrt{\kappa_k} T)} \Theta_k\left(\frac{T}{2}\right), & \text{if } \kappa_k > 0, \\ \frac{1}{2} \Theta_k\left(\frac{T}{2}\right), & \text{if } \kappa_k = 0. \end{cases}$$

$\Rightarrow$  Gradient descent to approximate the  $\text{prox}_{\lambda d_2}$  on  $\mathcal{M}^3$ .

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# A Signal of Cyclic Data

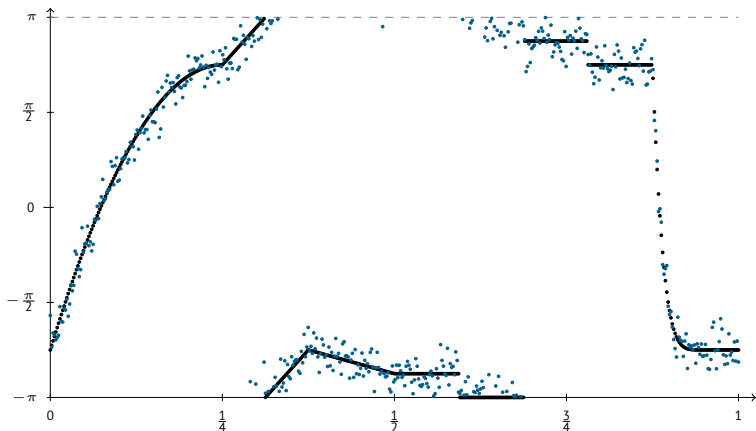
Noising a phase valued signal.



- function  $f: [0, 1] \rightarrow \mathbb{S}^1$  sampled to obtain data  $f_o = (f_{o,i})_{i=1}^{500}$
- $f$  could be a wrapped version of the gray plot, hence
- jumps  $> \pi$  at  $\frac{5}{16}$  and  $\frac{11}{16}$  are due to the representation system

# A Signal of Cyclic Data

Noising a phase valued signal.

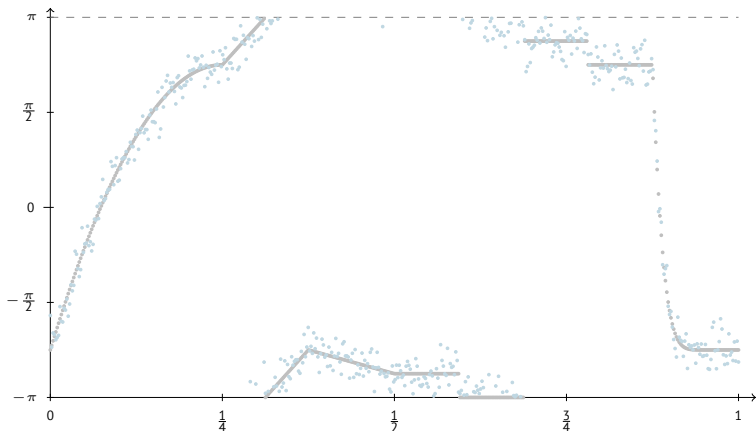


- function  $f: [0, 1] \rightarrow \mathbb{S}^1$  sampled to obtain data  $f_0 = (f_{0,i})_{i=1}^{500}$
- adding wrapped Gaussian noise,  $\sigma = 0.2$
- noisy data  $f_n = (f_0 + \eta)_{2\pi}$



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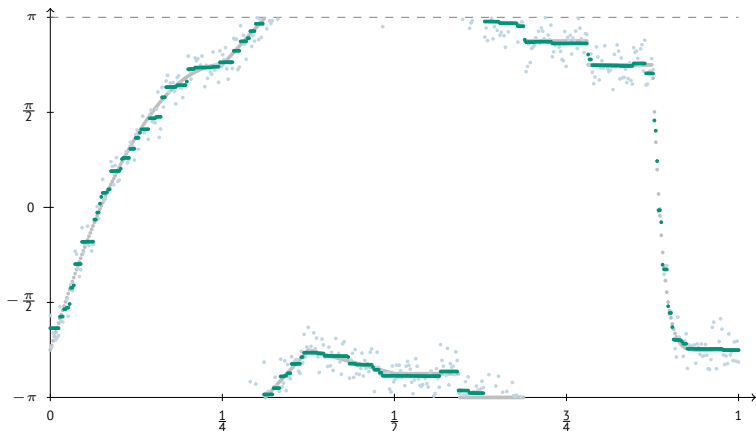
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■ comparison of  $f_0$  &  $f_n$  with

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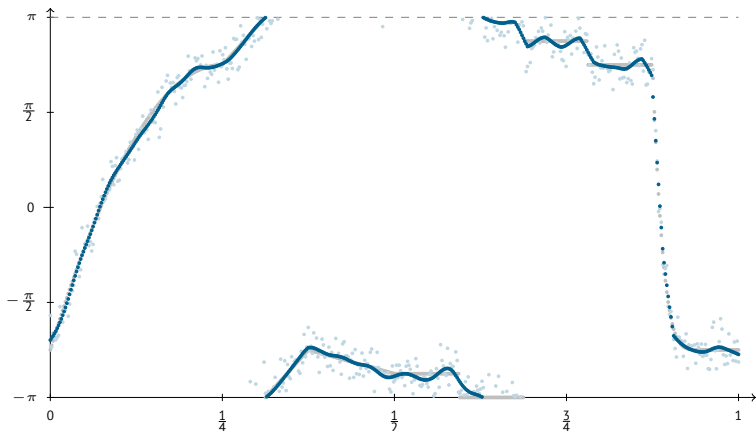
Denoising a phase valued signal.



- comparison of  $f_0$  &  $f_n$  with  $f_1$
- denoising  $TV_1$  ( $\alpha = \frac{3}{4}$ ,  $\beta = 0$ )
- but: stair casing

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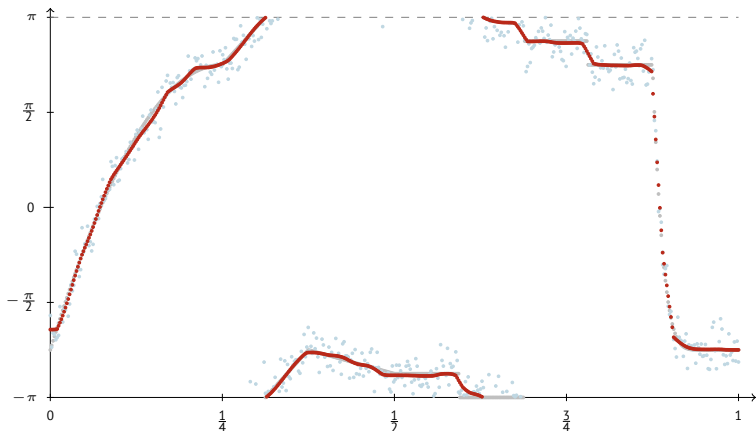
Denoising a phase valued signal.



- comparison of  $f_0$  &  $f_n$  with  $f_2$
- denoising  $TV_2$  ( $\alpha = 0, \beta = \frac{3}{2}$ )
- but: no plateaus

# A Signal of Cyclic Data

Denoising a phase valued signal.



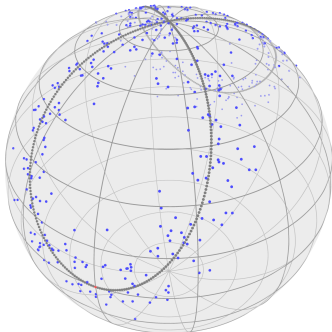
- comparison of  $f_0$  &  $f_n$  with  $f_3$
- denoising  $TV_1$  &  $TV_2$  ( $\alpha = \frac{1}{2}, \beta = 1$ )
- smallest mean squared error

# Bernoulli's Lemniscate on the Sphere $\mathbb{S}^2$

$$\gamma(t) := \frac{a\sqrt{2}}{\sin^2(t) + 1} (\cos(t), \cos 1 \sin(t))^T, \quad t \in [0, 2\pi], a = \frac{\pi}{2\sqrt{2}}.$$

Generate a **sphere-valued signal** by putting it into the tangential plane of the north pole

$$\gamma_{\mathbb{S}}(t) = \log_p(\gamma(t)), p = (0, 0, 1)^T$$



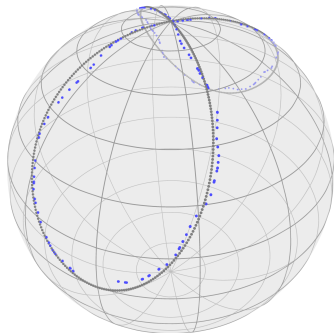
noisy lemniscate of Bernoulli on  $\mathbb{S}^2$ , Gaussian noise,  $\sigma = \frac{\pi}{30}$ , on  $T_p\mathbb{S}^2$ .

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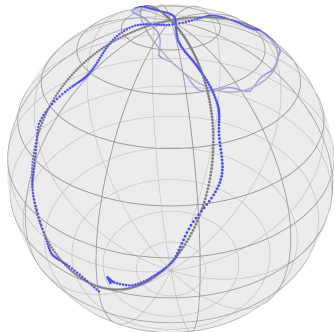
reconstruction with  $TV_1$ ,  $\alpha = 0.21$ ,  $MAE = 4.08 \times 10^{-2}$ .

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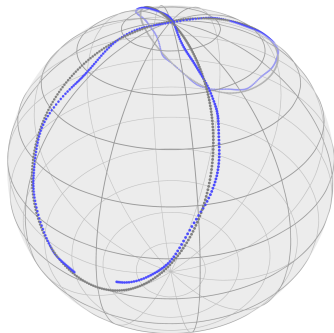
reconstruction with  $TV_2$ ,  $\alpha = 0$ ,  $\beta = 10$ ,  $MAE = 3.66 \times 10^{-2}$ .

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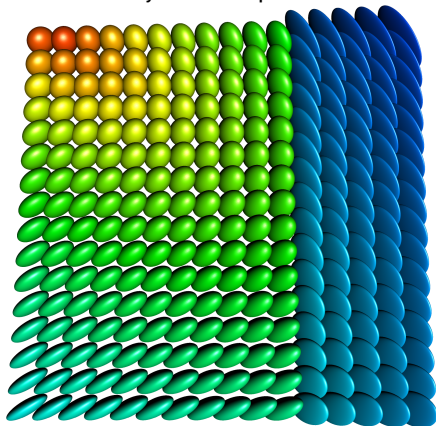


reconstruction with  $TV_1$  &  $TV_2$ ,  $\alpha = 0.16$ ,  $\beta = 12.4$ ,  $MAE = 3.27 \times 10^{-2}$ .



# Inpainting of $\mathcal{P}(3)$ -valued Images

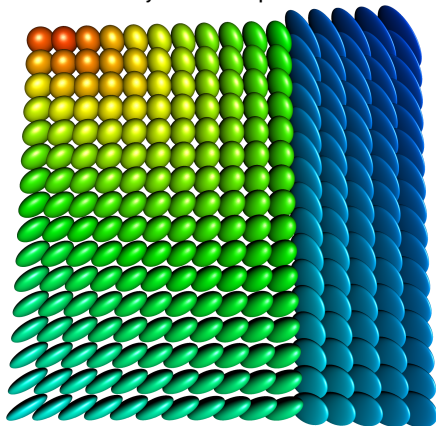
Illustrate symmetric positive definite  $3 \times 3$  matrices as ellipsoids



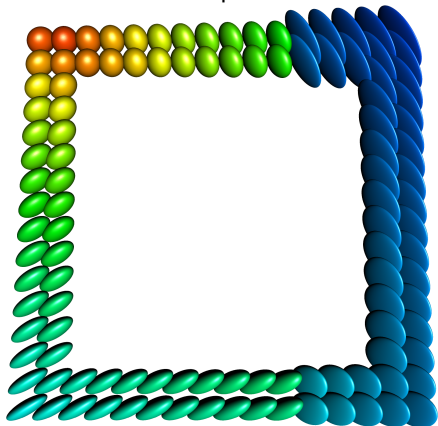
original data

# Inpainting of $\mathcal{P}(3)$ -valued Images

Illustrate symmetric positive definite  $3 \times 3$  matrices as ellipsoids



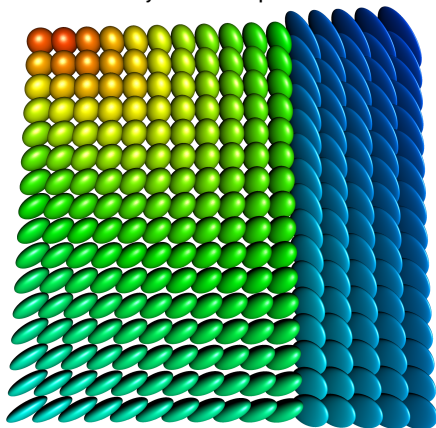
original data



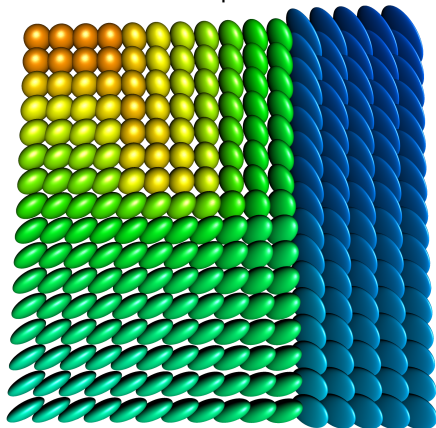
lost (a lot of) data

# Inpainting of $\mathcal{P}(3)$ -valued Images

Illustrate symmetric positive definite  $3 \times 3$  matrices as ellipsoids



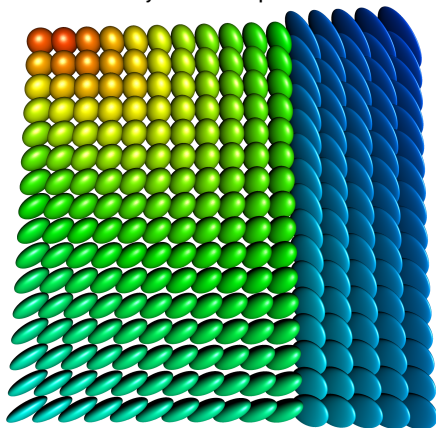
original data



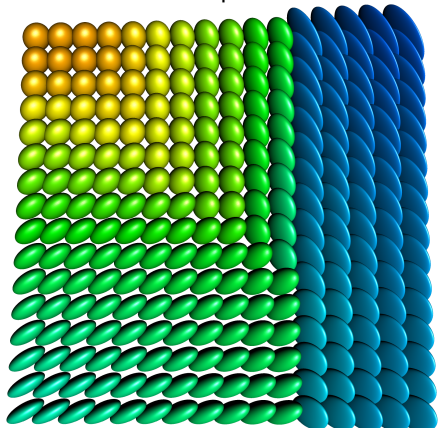
inpainted with  $\alpha = \beta = 0.05$ ,  
MAE = 0.0929

# Inpainting of $\mathcal{P}(3)$ -valued Images

Illustrate symmetric positive definite  $3 \times 3$  matrices as ellipsoids



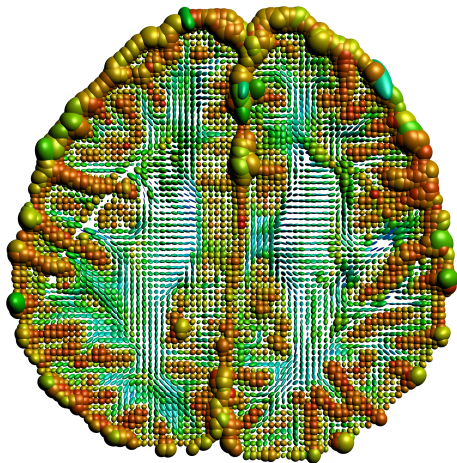
original data



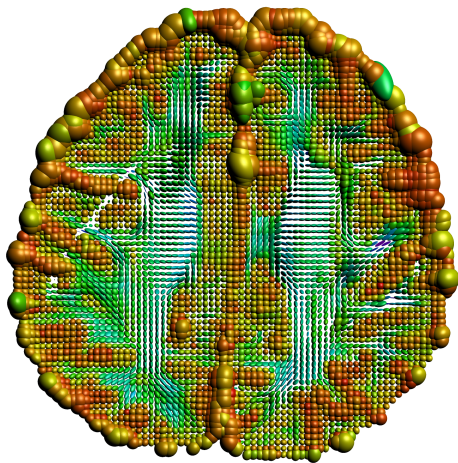
inpainted with  $\alpha = 0.1$ ,  
MAE = 0.0712

# Diffusion Tensor Magnetic Resonance Imaging

Dataset  $\mathcal{P}(3)^{112 \times 112 \times 50}$  of the human head, traversal plane  $z = 28$



original data,

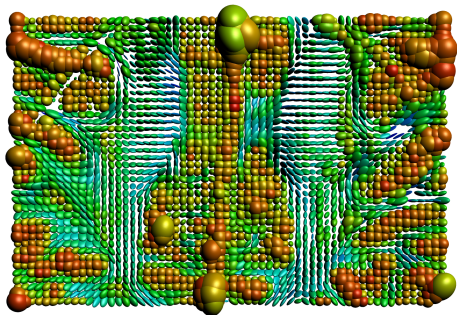


denoised,  $\alpha = 0.01$ ,  $\beta = 0.05$

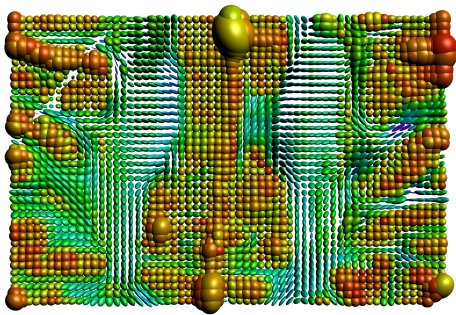
(Data available from The Camino Project, [cmic.cs.ucl.ac.uk/camino](http://cmic.cs.ucl.ac.uk/camino))

# Diffusion Tensor Magnetic Resonance Imaging

Dataset  $\mathcal{P}(3)^{112 \times 112 \times 50}$  of the human head, traversal plane  $z = 28$



original data,



denoised,  $\alpha = 0.01$ ,  $\beta = 0.05$

(Data available from The Camino Project, [cmic.cs.ucl.ac.uk/camino](http://cmic.cs.ucl.ac.uk/camino))

a detail

- 1 Introduction
- 2 Second Order Differences on Manifolds
- 3 Proximal Mappings and the Cyclic Proximal Point Algorithm
- 4 Examples
- 5 Conclusion**

# Conclusion

We have for manifold valued images  $f: \mathcal{V} \rightarrow \mathcal{M}$

- a model for a first and second order TV-type functional  $\mathcal{E}(u)$
- cyclic proximal point algorithm to minimize  $\mathcal{E}(u)$
- computed the proximal mappings using Jacobi fields
- proof of convergence

What's next?

- For manifold-valued TV: Parallel Douglas-Rachford Algorithm (J. Persch)

Future work

- an isotropic second order TV
- asymmetric differences
- other couplings, algorithms, manifolds,...



## Literature

- [1] M. Bačák, RB, G. Steidl, and A. Weinmann. A second order non-smooth variational model for restoring manifold-valued images. *SIAM J. Sci. Comput.*, accepted, 2016.
- [2] RB, F. Laus, G. Steidl, and A. Weinmann. Second order differences of cyclic data and applications in variational denoising. *SIAM J. Imaging Sci.*, 2014.
- [3] RB and A. Weinmann. A second order TV-type approach for inpainting and denoising higher dimensional combined cyclic and vector space data. *JMIV*, accepted, 2016.
- [4] J. Lellmann, E. Strelakovsky, S. Koetter, and D. Cremers. Total variation regularization for functions with values in a manifold. *IEEE ICCV*, 2013.
- [5] A. Weinmann, L. Demaret, and M. Storath. Total variation regularization for manifold-valued data. *SIAM J. Imaging Sci.*, 2014.

## Literature

- [1] M. Bačák, RB, G. Steidl, and A. Weinmann. A second order non-smooth variational model for restoring manifold-valued images. *SIAM J. Sci. Comput.*, accepted, 2016.
- [2] RB, F. Laus, G. Steidl, and A. Weinmann. Second order differences of cyclic data and applications in variational denoising. *SIAM J. Imaging Sci.*, 2014.
- [3] RB and A. Weinmann. A second order TV-type approach for inpainting and denoising higher dimensional combined cyclic and vector space data. *JMIV*, accepted, 2016.
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- [5] A. Weinmann, L. Demaret, and M. Storath. Total variation regularization for manifold-valued data. *SIAM J. Imaging Sci.*, 2014.

Thank you for your attention.