A Variational Model for Data Fitting on Manifolds by Minimizing the Acceleration of a Bézier Curve

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[a]joint work with P.-Y. Gousenbourger, UCLouvain, Louvain-la-Neuve, Belgium.

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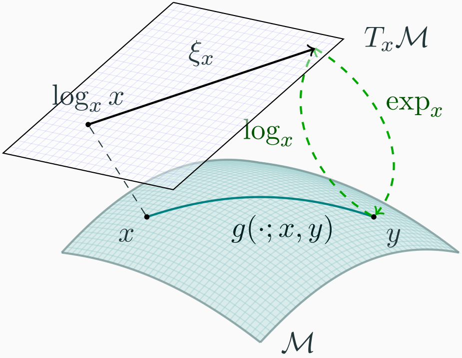
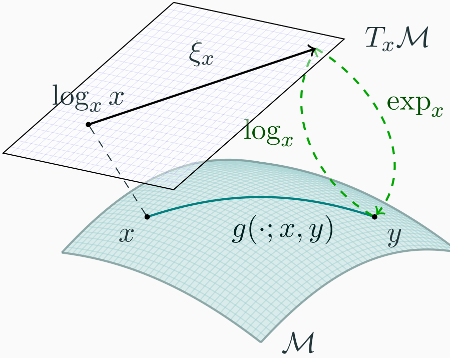
A d-dimensional Riemannian Manifold $\mathcal{M}$



A d-dimensional Riemannian manifold can be informally defined as a set $\mathcal{M}$ covered with a 'suitable' collection of charts, that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$ and a continously varying inner product on the tangent spaces.

[Absil, Mahony, Sepulchre, 2008]

A d-dimensional Riemannian Manifold $\mathcal{M}$



Geodesic $g(\cdot; x, y)$ shortest curve (on $\mathcal{M}$) between $x, y \in \mathcal{M}$
Tangent space $\mathrm{T}_x\mathcal{M}$ at x with inner product $\langle\cdot,\cdot\rangle_x$
Tangent bundle $\mathrm{T}\mathcal{M} \coloneqq \cup_{x\in\mathcal{M}}\mathrm{T}_x\mathcal{M}$
Logarithmic map $\log_x y \;= \dot{g}(0; x, y)$ "speed towards y"
Exponential map $\exp_x \xi_x = \gamma(1)$, where $\gamma(0) = x$, $\dot\gamma(0) = \xi_x$

Data Fitting on a Riemannian Manifold

Given data points $d_0, \ldots, d_n$ on a Riemannian manifold $\mathcal{M}$ and time points $t_i \in I$, find a "nice" curve $\gamma\colon I \to \mathcal{M}$, $\gamma \in \Gamma$, such that $\gamma(t_i) = d_i$ (interpolation) or $\gamma(t_i) \approx d_i$ (approximation).

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- Γ set of geodesics & approximation: geodesic regression
  [Rentmeesters, '11; Fletcher, '13; Boumal '13]
- Γ Sobolev space of curves: Inifinite-dimensional problem
  [Samir et. al.,'12]
- Γ composite Bézier curves; LSs in tangent spaces
  [Arnould et. al. '15; Gousenbourger, Massart, Absil, '18]
- Discretized curve, $\Gamma = \mathcal{M}^N$, [Boumal, Absil, '11]

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This talk

"nice" means minimal (discretized) acceleration ("as straight as possible") for Γ the set of composite Bézier curves.

In Euclidean space: Natural cubic splines as closed form solution.

Definition [Bézier, '62]

A Bézier curve β_K of degree $K \in \mathbb{N}_0$ is a function $\beta_K \colon [0,1] \to \mathbb{R}^d$ parametrized by control points $b_0, \ldots, b_K \in \mathbb{R}^n$ and defined by

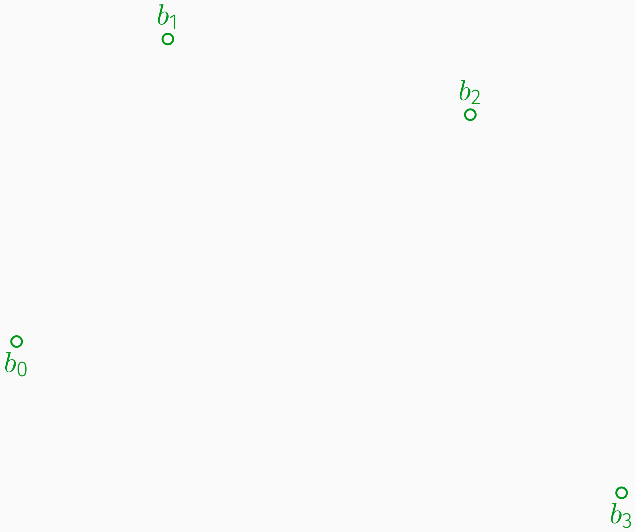
$$\beta_K(t; b_0, \ldots, b_K) := \sum_{j=0}^{K} b_j B_{j,K}(t),$$

[Bernstein, 1912]

where $B_{j,K} = \binom{K}{j} t^j (1-t)^{K-j}$ are the Bernstein polynomials of degree K.

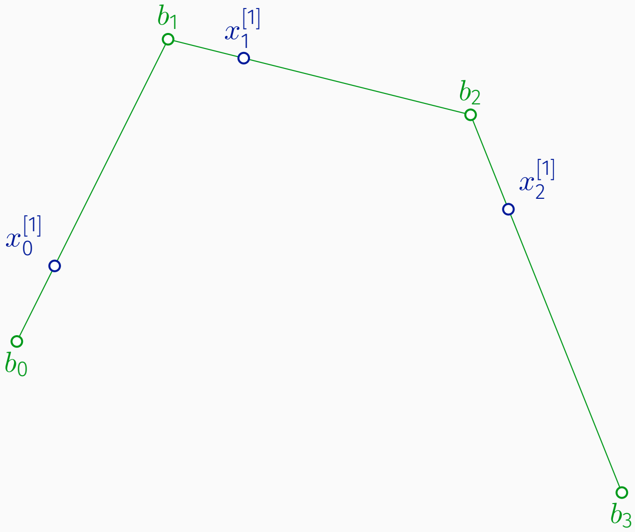
Evaluation via Casteljau's algorithm. [de Casteljau, '59]

Illustration of de Casteljau's Algorithm



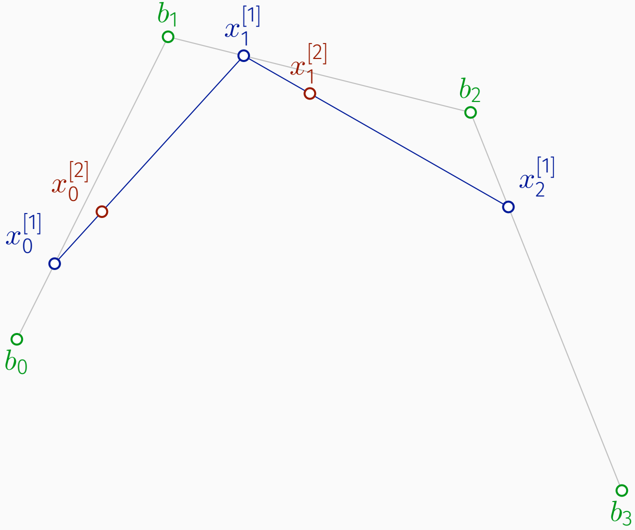
The set of control points b_0, b_1, b_2, b_3.

Illustration of de Casteljau's Algorithm



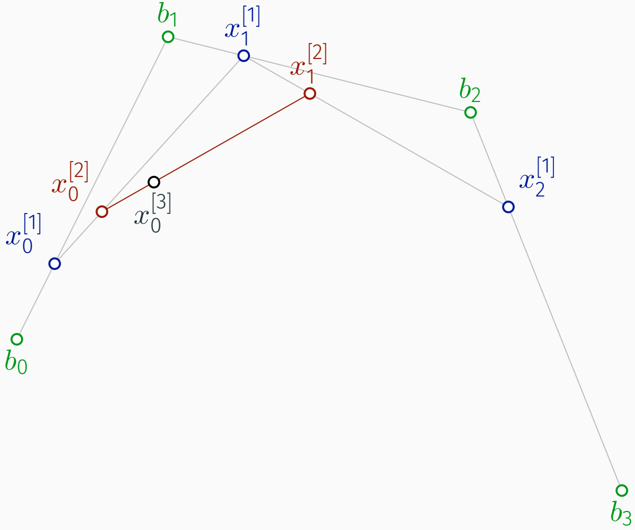
Evaluate line segments at $t = \frac{1}{4}$, obtain $x_0^{[1]}, x_1^{[1]}, x_2^{[1]}$.

Illustration of de Casteljau's Algorithm



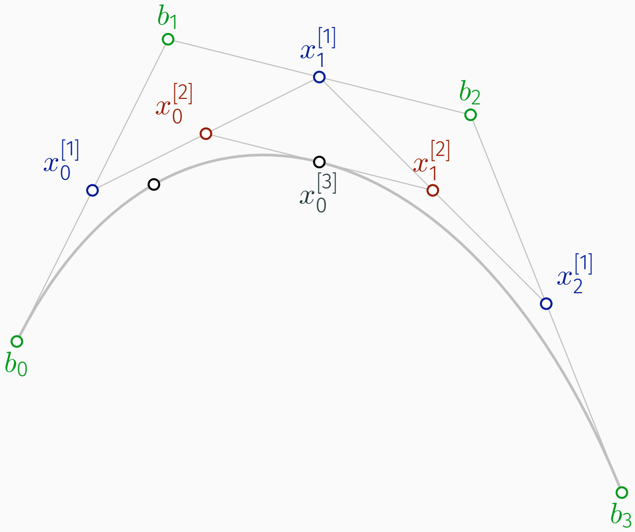
Repeat evaluation for new line segments to obtain $x_0^{[2]}, x_1^{[2]}$.

Illustration of de Casteljau's Algorithm



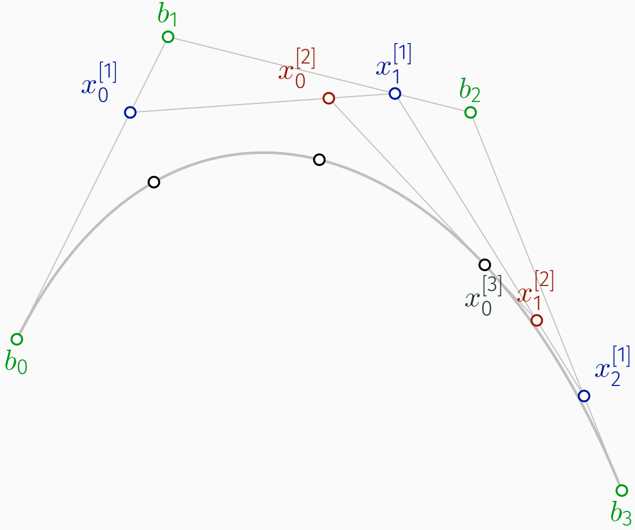
Repeat for the last segment to obtain $\beta_3(\frac{1}{4}; b_0, b_1, b_2, b_3) = x_0^{[3]}$.

Illustration of de Casteljau's Algorithm



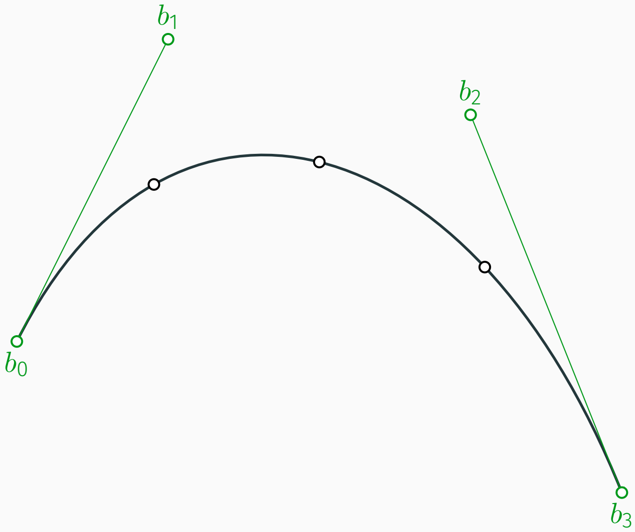
Same procedure for evaluation of $\beta_3(\frac{1}{2}; b_0, b_1, b_2, b_3)$.

Illustration of de Casteljau's Algorithm



Same procedure for evaluation of $\beta_3(\frac{3}{4}; b_0, b_1, b_2, b_3)$.

Illustration of de Casteljau's Algorithm



Complete curve $\beta_3(t; b_0, b_1, b_2, b_3)$.

Composite Bézier Curves

Definition
A composite Bezier curve $B\colon [0,n] \to \mathbb{R}^d$ is defined as

$$B(t) := \begin{cases} \beta_K(t; b_0^0, \ldots, b_K^0) & \text{if } t \in [0,1], \\ \beta_K(t-i; b_0^i, \ldots, b_K^i), & \text{if } t \in (i, i+1], \quad i = 1, \ldots, n-1. \end{cases}$$

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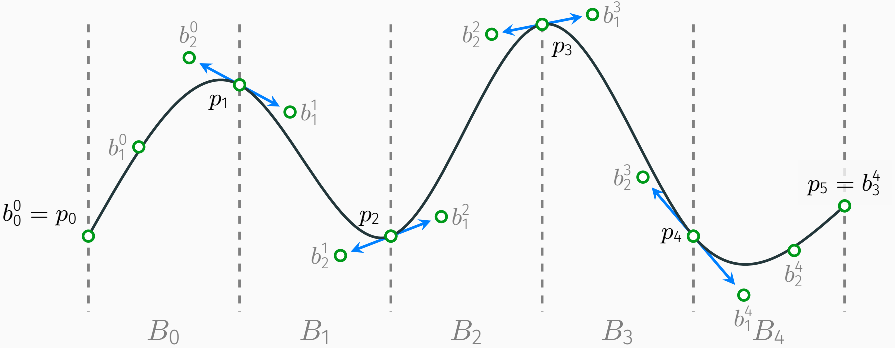
Denote ith segment by $B_i(t) = \beta_K(t; b_0^i, \ldots, b_K^i)$ and $p_i = b_0^i$.



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Denote ith segment by $B_i(t) = \beta_K(t; b_0^i, \dots, b_K^i)$ and $p_i = b_0^i$.

- continuous iff $B_{i-1}(1) = B_i(0)$, $i = 1, \dots, n-1$
  $\Rightarrow b_K^{i-1} = b_0^i = p_i$, $i = 1, \dots, n-1$
- continuously differentiable iff $p_i = \frac{1}{2}(b_{K-1}^{i-1} + b_1^i)$
  $\Rightarrow$ written with connecting line segment $p_i = g(\frac{1}{2}, b_{K-1}^{i-1}, b_1^i)$

Riemannian Bézier Curves

Definition. [Park, Ravani, 1995; Popiel, Noakes, 2007]
Let $\mathcal{M}$ be a Riemannian manifold and $b_0, \ldots, b_K \in \mathcal{M}$, $K \in \mathbb{N}$.

The (generalized) Bézier curve of degree k, $k \leq K$, is defined as

$$\beta_k(t; b_0, \ldots, b_k) = g(t; \beta_{k-1}(t; b_0, \ldots, b_{k-1}), \beta_{k-1}(t; b_1, \ldots, b_k)),$$

if $k > 0$, and

$$\beta_0(t; b_0) = b_0.$$

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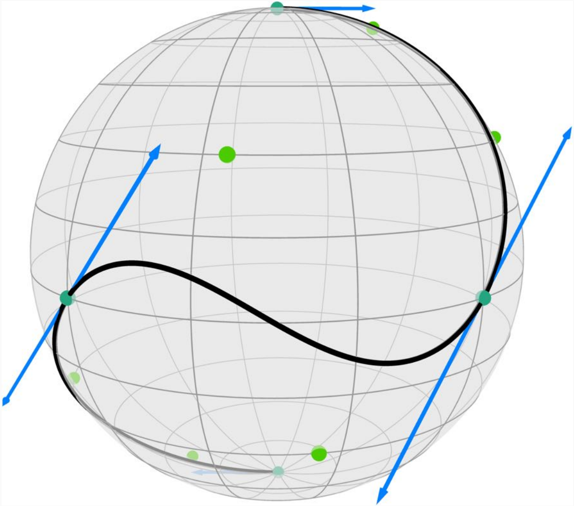
- Bézier curves $\beta_1(t; b_0, b_1) = g(t; b_0, b_1)$ are geodesics.
- composite Bézier curves $B\colon [0, n] \to \mathcal{M}$ completely analogue (using geodesics for line segments)

Illustration of a Composite Bezier Curve on the Sphere



The directions, e.g. $\log_{p_j} b_j^1$, are now tangent vectors.

A Variational Model for Data Fitting

Let $d_0, \ldots, d_n \in \mathcal{M}$. A model for data fitting reads

$$E(B) = \frac{\lambda}{2} \sum_{k=0}^{n} d^2_{\mathcal{M}}(B(k), d_k) + \int_0^n \Big\| \frac{D^2 B(t)}{\mathrm{d}t^2} \Big\|^2_{B(t)}, \mathrm{d}t \qquad \lambda > 0,$$

where $B \in \Gamma$ is from the set of continuously differentiable composite Bezier curve of degree K with n segments.

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$$E(B) = \frac{\lambda}{2} \sum_{k=0}^{n} d_{\mathcal{M}}^2(B(k), d_k) + \int_0^n \left\| \frac{D^2 B(t)}{\mathrm{d}t^2} \right\|_{B(t)}^2, \mathrm{d}t \qquad \lambda > 0,$$

where $B \in \Gamma$ is from the set of continuously differentiable composite Bezier curve of degree K with n segments.

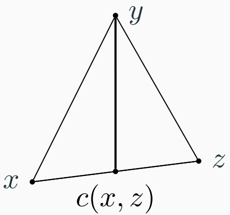
- Goal: find minimizer $B^* \in \Gamma$
- finite dimensional optimization problem (in variables b_j^i) on $\mathcal{M}^L$ with
  - $L = n(K-1) + 2$
  - $\lambda \to \infty$ yields interpolation ($p_k = d_k$) $\Rightarrow L = n(K-2) + 1$

Second order difference: [RB et al., 2014; RB, Weinmann, 2016; Bačák et al., 2016]

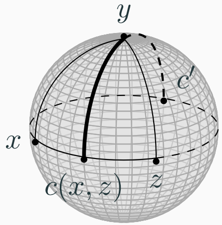
$$\mathrm{d}_2(x,y,z) \coloneqq \min_{c \in \mathcal{C}_{x,z}} \mathrm{d}_{\mathcal{M}}(c,y), \quad x,y,z \in \mathcal{M},$$

$\mathcal{C}_{x,z}$ mid point(s) of geodesic(s) $g(\cdot;x,z)$

$\frac{1}{2}\|x-2y+z\|_2 = \|\frac{1}{2}(x+z)-y\|_2$



$\min_{c \in \mathcal{C}_{x,z}} d_{\mathcal{M}}(c,y)$



$\mathcal{M} = \mathbb{S}^2$

Discretizing the Data Fitting Model

We discretize the absolute second order covariant derivative

$$\int_0^n \left\| \frac{D^2 B(t)}{\mathrm{d}t^2} \right\|^2_{\gamma(t)} \mathrm{d}t \approx \sum_{k=1}^{N-1} \frac{\Delta_s d_2^2[B(s_{i-1}), B(s_i), B(s_{i+1})]}{\Delta_s^4}.$$

on equidistant points $s_0, \ldots, s_N$ with step size $\Delta_s = s_1 - s_0$.

Evaluating $E(B)$ consists of evaluation of geodesics and squared (Riemannian) distances

- $(N+1)K$ geodesics to evaluate the Bézier segments
- N geodesics to evaluate the mid points
- N distances to obtain the second order absolute finite differences squared

Gradient and Chain Rule on a Manifold

The gradient $\nabla_{\mathcal{M}} f(x) \in T_x\mathcal{M}$ of $f\colon \mathcal{M} \to \mathbb{R}$, $x \in \mathcal{M}$, is defined as the tangent vector that fulfills

$$\langle \nabla_{\mathcal{M}} f(x), \xi \rangle_x = D_x f[\xi] \quad \text{for all} \quad \xi \in T_x\mathcal{M}.$$

For a composition $F(x) = (g \circ h)(x) = g(h(x))$ of two functions $g, h\colon \mathcal{M} \to \mathcal{M}$ the chain rule reads for $x \in \mathcal{M}$ and $\xi \in T_x\mathcal{M}$ as

$$D_x F[\xi] = D_{h(x)} g\bigl[D_x h[\xi]\bigr],$$

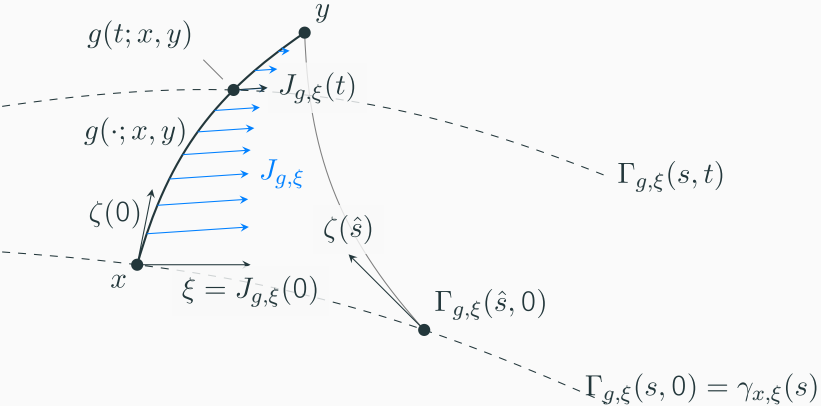
where $D_x h[\xi] \in T_{h(x)}\mathcal{M}$ and $D_x F[\xi] \in T_{F(x)}\mathcal{M}$.

The Differential of a Geodesic w.r.t. its Start Point

The geodesic variation

$$\Gamma_{g,\xi}(s,t) := \exp_{\gamma_{x,\xi}(s)}(t\zeta(s)), \qquad s \in (-\varepsilon,\varepsilon),\ t \in [0,1], \varepsilon > 0.$$

is used to define the Jacobi field $J_{g,\xi}(t) = \frac{\partial}{\partial s}\Gamma_{g,\xi}(s,t)|_{s=0}$.



Then the differential reads $D_x g(t,\cdot,y)[\xi] = J_{g,\xi}(t)$.

Implementing Jacobi Fields

- On symmetric manifolds, the Jacobi field can be evaluated in closed form, since the PDE decouples into ODEs.
- The adjoint Jacobi fields $J^*_{g,\eta}(t)$ are characterized by

$$\langle J_{g,\xi}(t), \eta \rangle_{g(t)} = \langle \xi, J^*_{g,\eta}(t) \rangle_x, \quad \text{for all } \xi \in T_x\mathcal{M}, \eta \in T_{g(t;x,y)}\mathcal{M}$$

  can be computed without extra efforts, i.e. the same ODEs occur.

$\Rightarrow$ adjoint Jacobi fields can be used to calculate the gradient

- Gradient of iterated evaluations of geodesics can be computed by composition of (adjoint) Jacobi fields

Gradient Descent on a Riemannian Manifold

Let $\mathcal{N} = \mathcal{M}^L$ be the product manifold of $\mathcal{M}$,

Input.

- $f\colon \mathcal{N} \to \mathbb{R}$,
- its gradient $\nabla_{\mathcal{N}} f$,
- initial data $x^{(0)} = b \in \mathcal{N}$
- step sizes $s_k > 0, k \in \mathbb{N}$.

Output: $\hat{x} \in \mathcal{N}$

$k \leftarrow 0$

repeat

$\quad x^{(k+1)} \leftarrow \exp_{x^{(k)}}\bigl(-s_k \nabla_{\mathcal{N}} f(x^{(k)})\bigr)$

$\quad k \leftarrow k + 1$

until a stopping criterion is reached

return $\hat{x} := x^{(k)}$

Armijo Step Size Rule

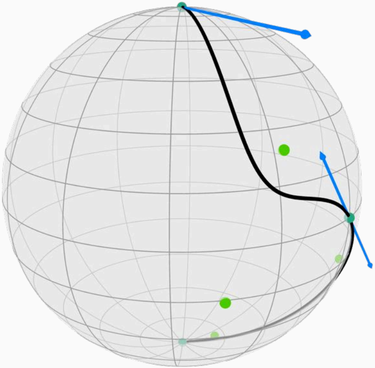
Let $x = x^{(k)}$ be an iterate from the gradient descent algorithm, $\beta, \sigma \in (0, 1), \alpha > 0$.

Let m be the smallest positive integer such that

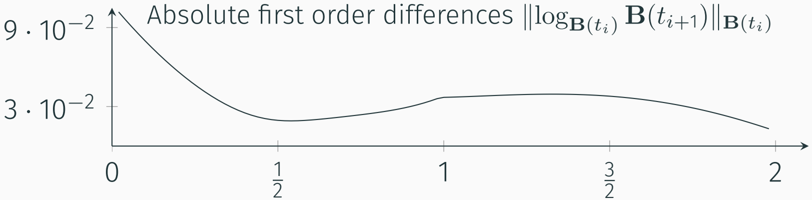
$$f(x) - f\big(\exp_x(-\beta^m \alpha \nabla_{\mathcal{N}} f(x))\big) \geq \sigma \beta^m \alpha \|\nabla_{\mathcal{N}} f(x)\|_x$$

Set the step size $s_k \coloneqq \beta^m \alpha$.

Minimizing with Known Minimizer



Original



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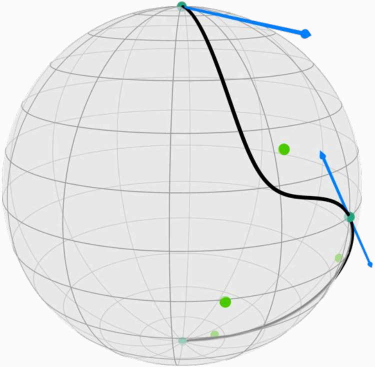

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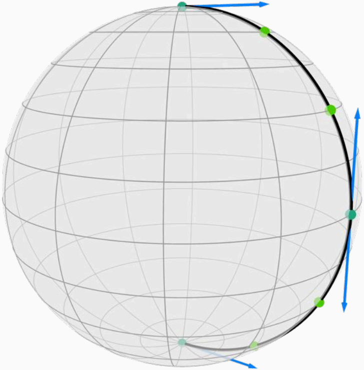
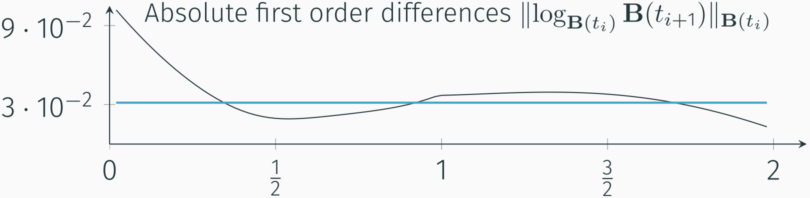
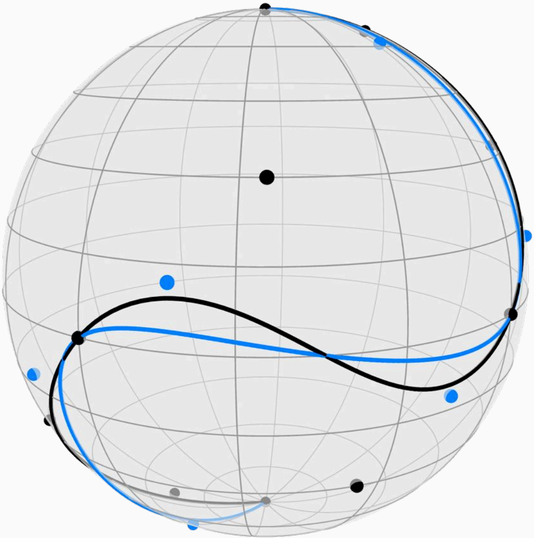


Minimized



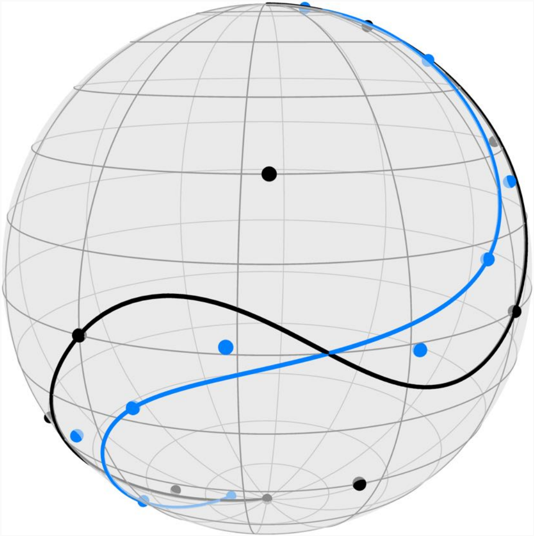
Interpolation by Bezier Curves with Minimal Acceleration.



A comp. Bezier curve (black) and its mnimizer (blue).

Approximation by Bezier Curves with Minimal Acceleration.

In the following video λ is slowly decreased from 10 to 0.



The initial setting, $\lambda = 10$.

Approximation by Bezier Curves with Minimal Acceleration.

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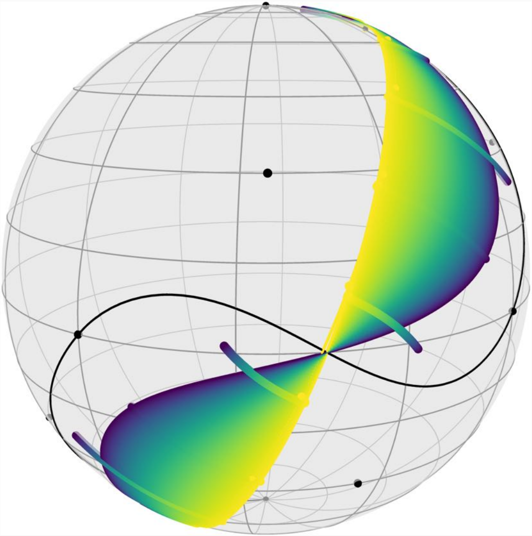
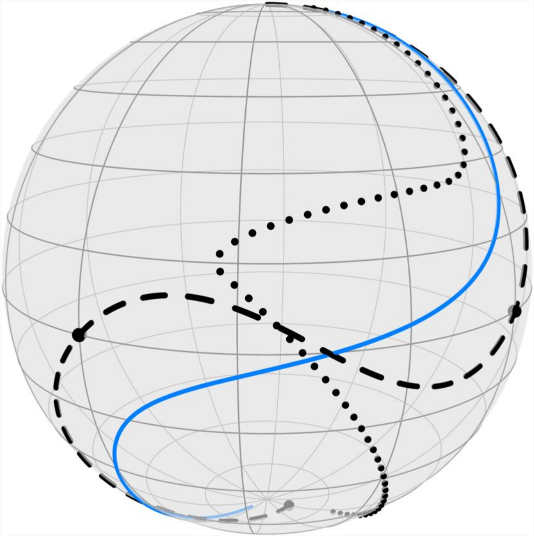


Summary of the video.

Comparison to Previous Approach

[Gousenbourger, Massart, Absil, 2018]

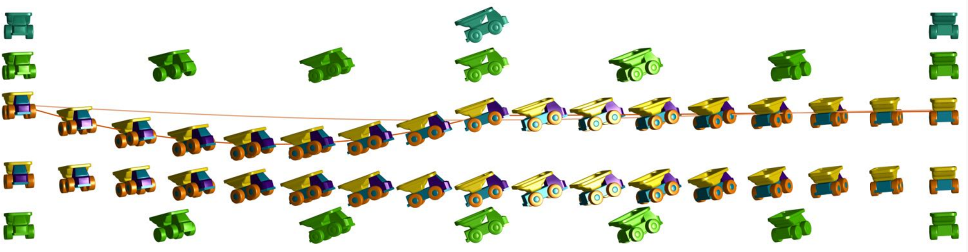


This curve (dashed) is “too global” to be solved in a tangent space (dotted) correctly, while our method (blue) still works.

An Example of Rotations $\mathcal{M} = \mathrm{SO}(3)$

Initialization with approach from composite splines

[Gousenbourger, Massart, Absil, 2018]



Our method outperforms the approach of solving linear systems in tangent spaces, but their approach can serve as an initialization.

Conclusion & Literature

- Data fitting on manifolds with Bézier curves minimizing their acceleration
- computed the gradient with respect to control points
- implemented within the MVIR toolbox (available soon)

ronnybergmann.net/mvirt/

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Literature

R. Bergmann and P.-Y. Gousenbourger. *A variational model for data fitting on manifolds by minimizing the acceleration of a Bézier curve.* 2018. arXiv: 1807.10090.

P.-Y. Gousenbourger, E. Massart, and P.-A. Absil. *Data fitting on manifolds with composite Bézier-like curves and blended cubic splines.* submitted. 2018.

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