A Primal-Dual Algorithm for Convex Nonsmooth Optimization on Riemannian Manifolds

Ronny Bergmann

joint work with
Roland Herzog, Maurício Silva Louzeiro, Daniel Tenbrinck, José Vidal-Núñez.

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Norwegian University of Science and Technology

The Model

We consider a minimization problem

$$\operatorname*{arg\,min}_{p \in \mathcal{C}} F(p) + G(\Lambda(p))$$

- $\mathcal{M}, \mathcal{N}$ are (high-dimensional) Riemannian Manifolds
- $F\colon \mathcal{M} \to \overline{\mathbb{R}}$ nonsmooth, (locally, geodesically) convex
- $G\colon \mathcal{N} \to \overline{\mathbb{R}}$ nonsmooth, (locally) convex
- $\Lambda\colon \mathcal{M} \to \mathcal{N}$ nonlinear
- $\mathcal{C} \subset \mathcal{M}$ strongly geodesically convex.

➜ In image processing:
choose a model, such that finding a minimizer yields the reconstruction

Splitting Methods & Algorithms

On a Riemannian manifold $\mathcal{M}$ we have

- Cyclic Proximal Point Algorithm (CPPA) [Bačák 2014]
- (parallel) Douglas–Rachford Algorithm (PDRA) [RB, Persch, and Steidl 2016]

On $\mathbb{R}^n$ PDRA is known to be equivalent to [O'Connor and Vandenberghe 2018; Setzer 2011]

- Primal-Dual Hybrid Gradient Algorithm (PDHGA) [Esser, Zhang, and Chan 2010]
- Chambolle-Pock Algorithm (CPA) [Chambolle and Pock 2011; Pock, Cremers, Bischof, and Chambolle 2009]

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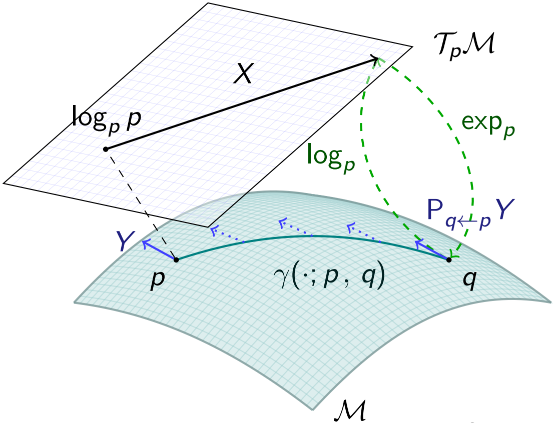
Goals of this talk.

Formulate Duality on a Manifold

Derive a Riemannian Chambolle–Pock Algorithm (RCPA)

A d-dimensional Riemannian manifold $\mathcal{M}$



A d-dimensional Riemannian manifold can be informally defined as a set $\mathcal{M}$ covered with a 'suitable' collection of charts, that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$ and a continuously varying inner product on the tangent spaces.

[Absil, Mahony, and Sepulchre 2008]

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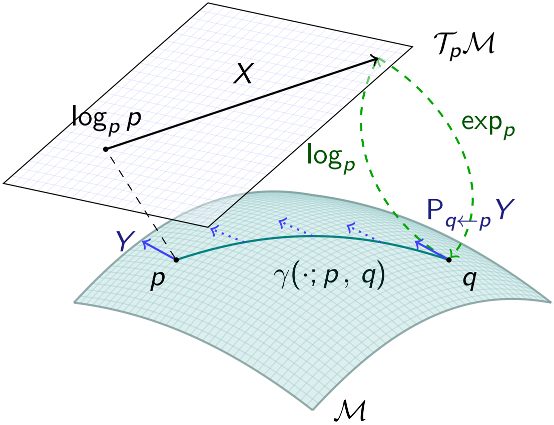

Geodesic $\gamma(\cdot;p,q)$
: a shortest path between $p, q \in \mathcal{M}$

Tangent space $\mathcal{T}_p\mathcal{M}$ at p
: with inner product $(\cdot,\cdot)_p$

Logarithmic map $\log_p q = \dot\gamma(0;p,q)$
: "speed towards q"

Exponential map $\exp_p X = \gamma_{p,X}(1)$,
: where $\gamma_{p,X}(0) = p$ and $\dot\gamma_{p,X}(0) = X$

Parallel transport $\mathrm{P}_{q\leftarrow p}Y$
: from $\mathcal{T}_p\mathcal{M}$ along $\gamma(\cdot;p,q)$ to $\mathcal{T}_q\mathcal{M}$

The Euclidean Fenchel Conjugate

We define the Fenchel conjugate $f^*\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ by

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- interpretation: maximize the distance of $\xi^{\mathsf{T}}x$ to f

$\Rightarrow$ extremum seeking problem on the epigraph

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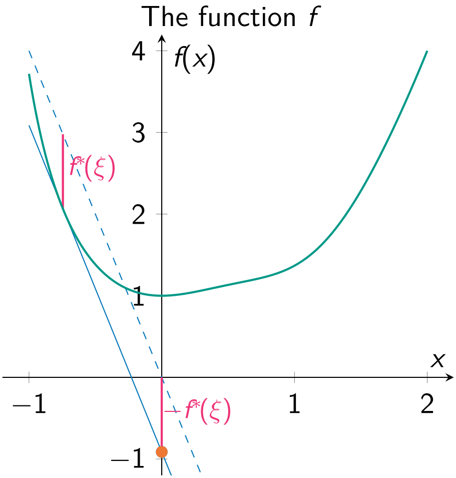
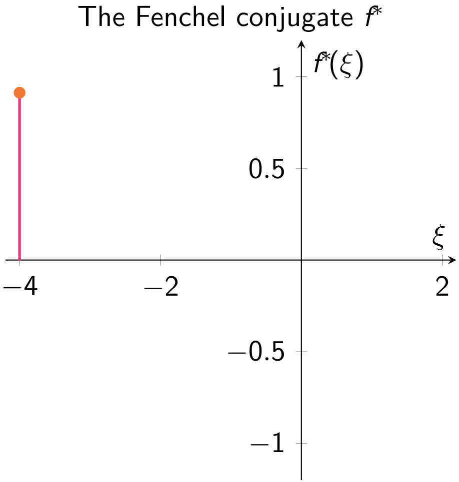
The Fenchel biconjugate reads

$$f^{**}(x) = (f^*)^*(x) = \sup_{\xi\in\mathbb{R}^n} \{\langle \xi , x\rangle - f^*(\xi)\}.$$

Illustration of the Fenchel Conjugate





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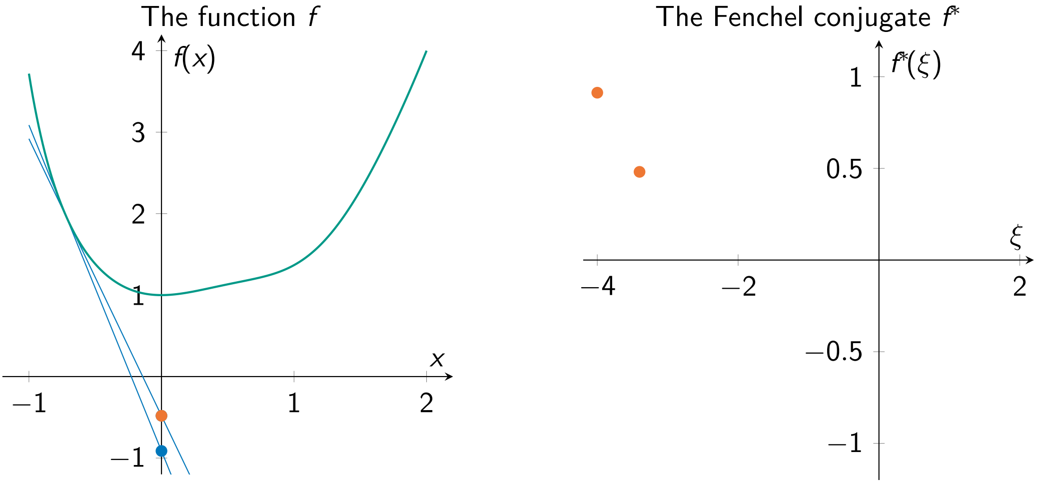

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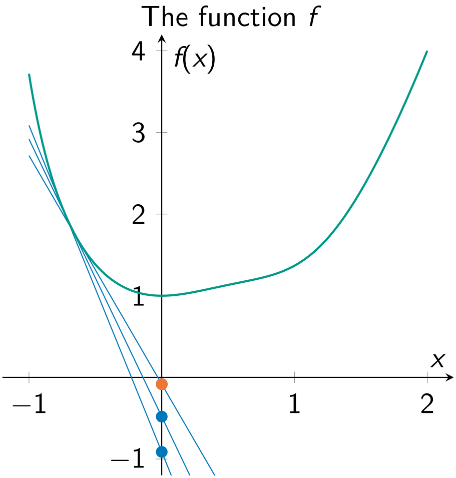

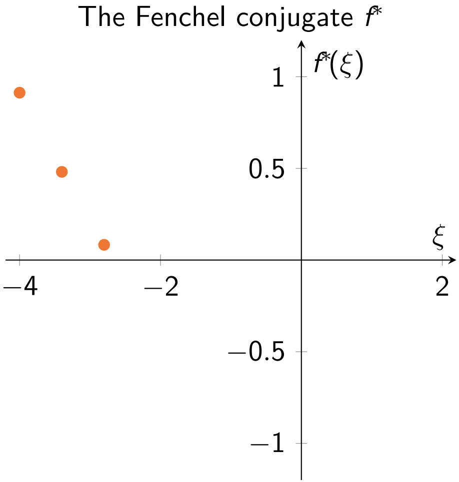

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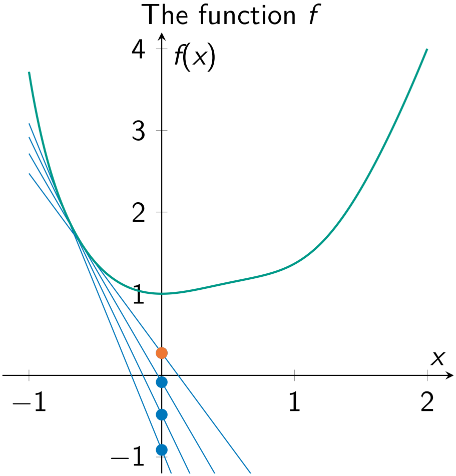

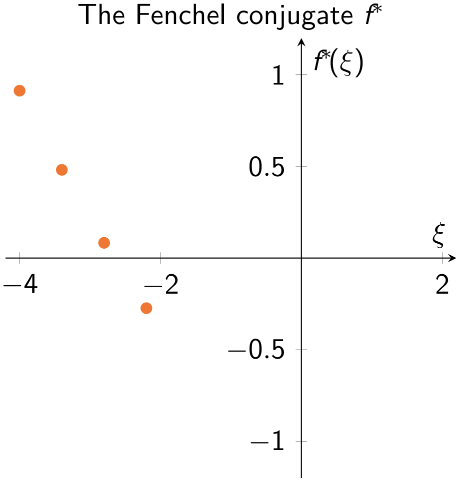

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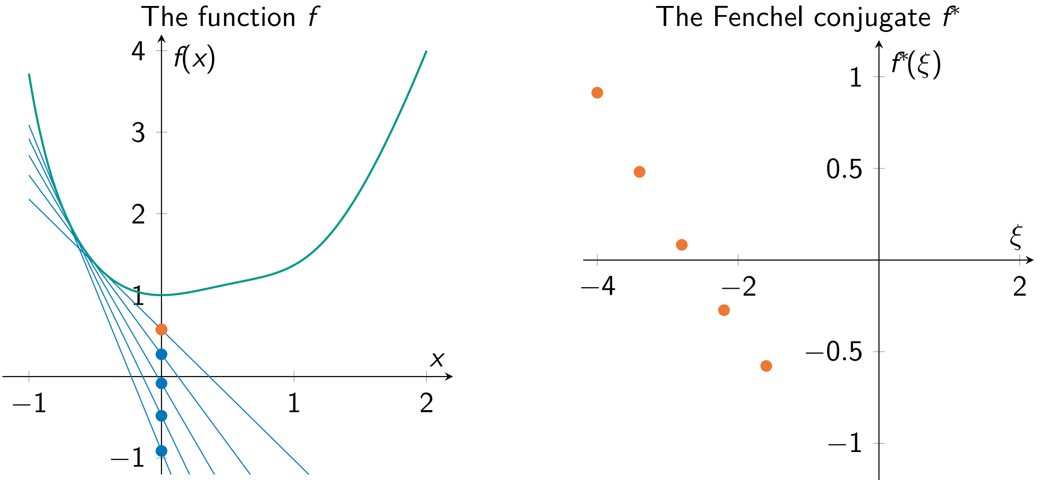

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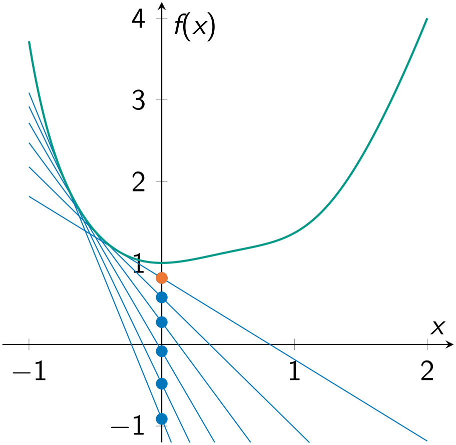
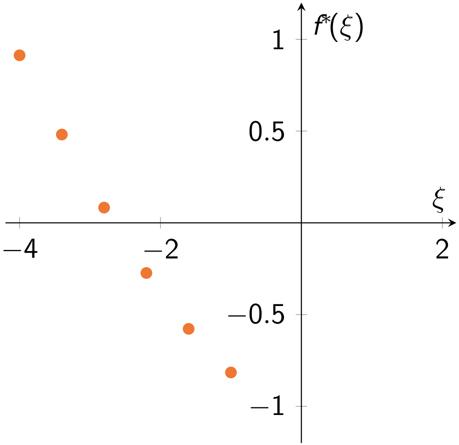
The function f



The Fenchel conjugate f^*



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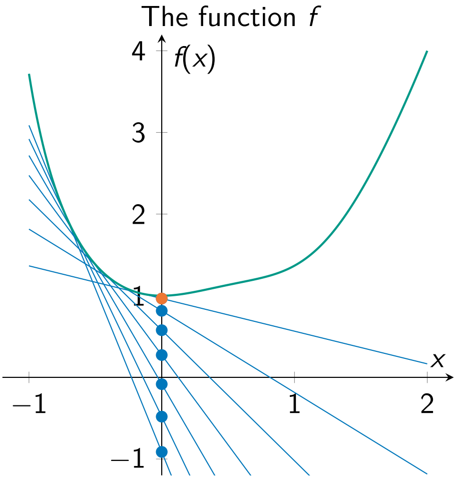

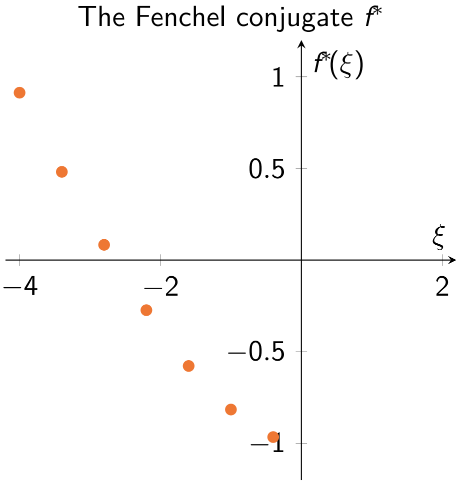

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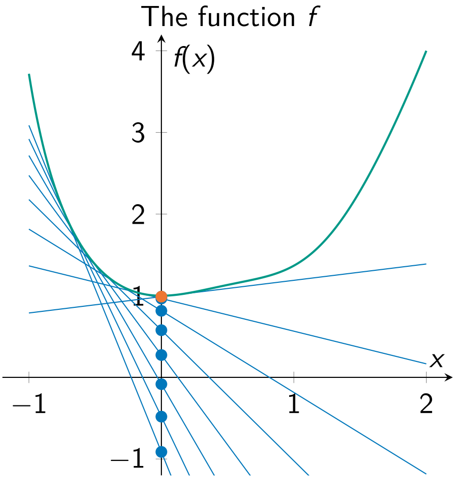

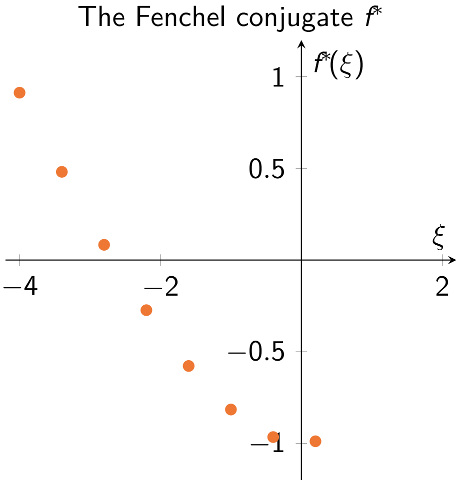

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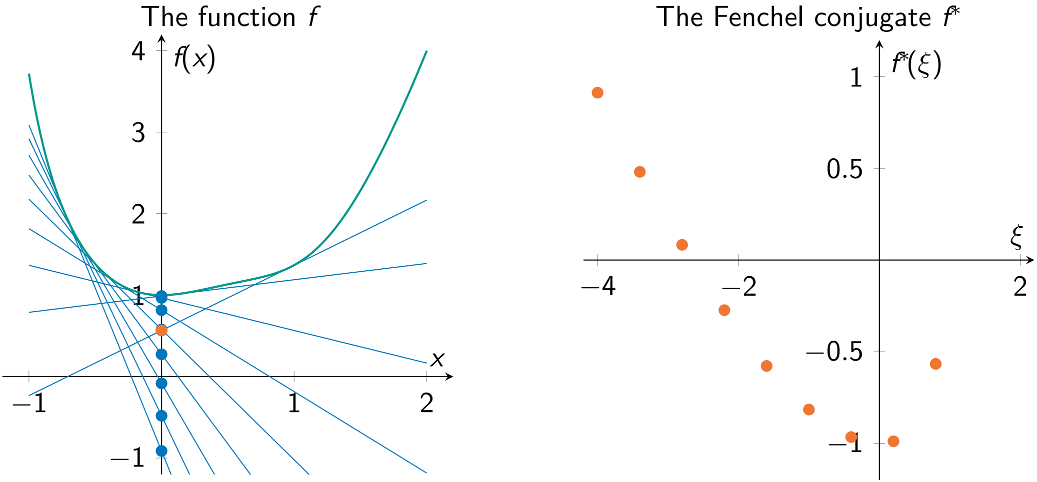

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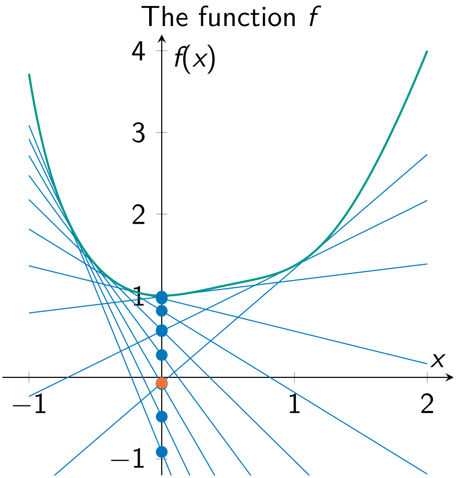

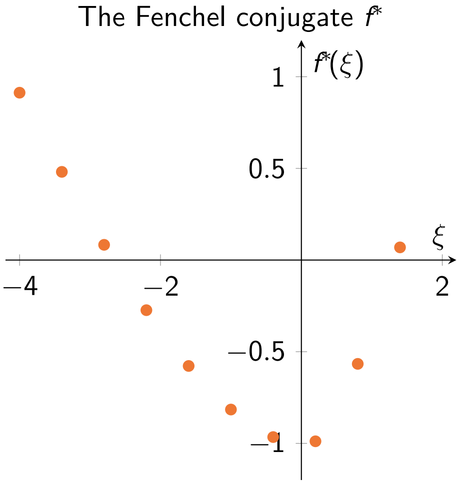

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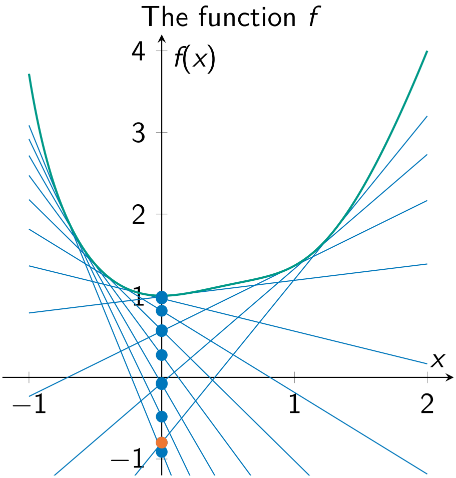

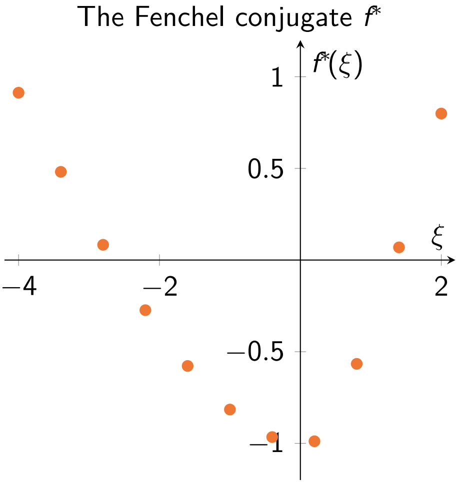

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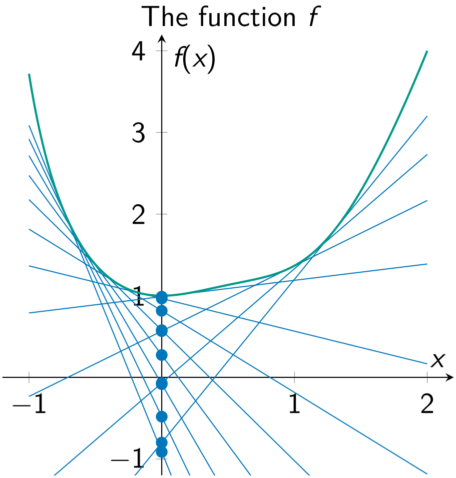

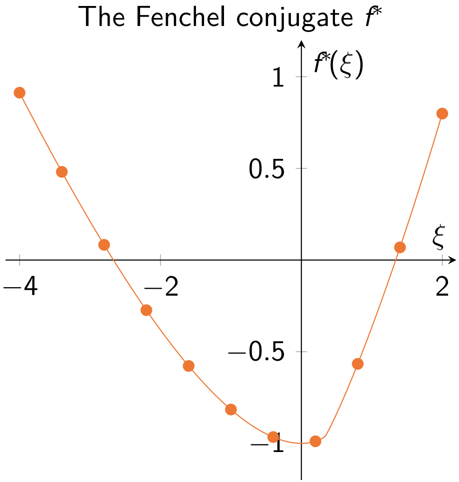

The Riemannian $m-$Fenchel Conjugate

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]
alternative approach: [Ahmadi Kakavandi and Amini 2010]

Idea: Introduce a point on $\mathcal{M}$ to “act as” 0.

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Idea: Introduce a point on $\mathcal{M}$ to "act as" 0.
Let $m \in \mathcal{C} \subset \mathcal{M}$ be given and $F\colon \mathcal{C} \to \overline{\mathbb{R}}$.
The m-Fenchel conjugate $F_m^*\colon \mathcal{T}_m^*\mathcal{M} \to \overline{\mathbb{R}}$ is defined by

$$F_m^*(\xi_m) \coloneqq \sup_{X \in \mathcal{L}_{\mathcal{C},m}} \bigl\{ \langle \xi_m , X \rangle - F(\exp_m X) \bigr\},$$

where $\mathcal{L}_{\mathcal{C},m} \coloneqq \{ X \in \mathcal{T}_m\mathcal{M} \mid q = \exp_m X \in \mathcal{C} \text{ and } \|X\|_p = d(q,p) \}$.

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Let $m' \in \mathcal{C}$.

The mm'-Fenchel-biconjugate $F_{mm'}^{**}\colon \mathcal{C} \to \overline{\mathbb{R}}$ is given by

$$F_{mm'}^{**}(p) = \sup_{\xi_{m'} \in \mathcal{T}_{m'}^*\mathcal{M}} \bigl\{\langle \xi_{m'} , \log_{m'} p\rangle - F_m^*(\mathrm{P}_{m \leftarrow m'}\xi_{m'})\bigr\}.$$

usually we only use the case $m = m'$.

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Saddle Point Formulation

Let F be geodesically convex, $G \circ \exp_n$ be convex (on $\mathcal{T}_n\mathcal{N}$).

From

$$\min_{p\in\mathcal{C}} F(p) + G(\Lambda(p))$$

we derive the saddle point formulation for the n-Fenchel conjugate of G as

$$\min_{p\in\mathcal{C}} \max_{\xi_n\in\mathcal{T}_n^*\mathcal{N}} \langle\xi_n\,, \log_n \Lambda(p)\rangle + F(p) - G_n^*(\xi_n).$$

But $\Lambda\colon \mathcal{M} \to \mathcal{N}$ is a non-linear operator!

For Optimality Conditions and the Dual Prolem: What's Λ^*?

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[Valkonen 2014]

Approach. Linearization: $\Lambda(p) \approx \exp_{\Lambda(m)} D\Lambda(m)[\log_m p]$

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The Exact Riemannian Chambolle–Pock Algorithm (eRCPA)

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

Input: $p^{(0)} \in \mathbb{R}^d$, $\xi^{(0)} \in \mathbb{R}^d$, and parameters σ, τ, $\theta > 0$

1: $k \leftarrow 0$
2: $\bar{p}^{(0)} \leftarrow p^{(0)}$
3: **while** not converged **do**
4: $\quad \xi^{(k+1)} \leftarrow \operatorname{prox}_{\tau G^*}\big(\xi^{(k)} + \tau(\quad \Lambda(\bar{p}^{(k)})) \ \big)$
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5: $\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\Bigl(\exp_{p^{(k)}}\Bigl(\mathrm{P}_{p^{(k)}\leftarrow m}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)\Bigr)$

6: $\quad \bar{p}^{(k+1)} \leftarrow p^{(k+1)} - \theta(p^{(k)} - p^{(k+1)})$

7: $\quad k \leftarrow k + 1$

8: **end while**

Output: $p^{(k)}$

The Exact Riemannian Chambolle–Pock Algorithm (eRCPA)

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

Input: m, $p^{(0)} \in \mathcal{C} \subset \mathcal{M}$, $n = \Lambda(m)$, $\xi_n^{(0)} \in \mathcal{T}_n^*\mathcal{N}$, and parameters σ, τ, $\theta > 0$

1: $k \leftarrow 0$

2: $\bar{p}^{(0)} \leftarrow p^{(0)}$

3: **while** not converged **do**

4: $\quad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau G_n^*}\Big(\xi_n^{(k)} + \tau\big(\log_n \Lambda(\bar{p}^{(k)})\big)^\flat\Big)$

5: $\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\Bigg(\exp_{p^{(k)}}\Big(\mathrm{P}_{p^{(k)} \leftarrow m}\big(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\big)^\sharp\Big)\Bigg)$

6: $\quad \bar{p}^{(k+1)} \leftarrow \exp_{p^{(k+1)}}\big(-\theta \log_{p^{(k+1)}} p^{(k)}\big)$

7: $\quad k \leftarrow k + 1$

8: **end while**

Output: $p^{(k)}$

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Generalizations & Variants of the RCPA

Classically

[Chambolle and Pock 2011]

- change $\sigma = \sigma_k$, $\tau = \tau_k$, $\theta = \theta_k$ during the iterations
- introduce an acceleration γ
- relax dual $\bar{\xi}$ instead of primal $\bar{p}$ (switches lines 4 and 5)

Furthermore we

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

- introduce the IRCPA: linearize Λ, i. e., adopt the Euclidean case from [Valkonen 2014]

$$\log_n \Lambda(\bar{p}^{(k)}) \quad \to \quad \mathrm{P}_{n \leftarrow \Lambda(m)} D\Lambda(m)[\log_m \bar{p}^{(k)}]$$

- choose $n \neq \Lambda(m)$ introduces a parallel transport

$$D\Lambda(m)^*[\xi_n^{(k+1)}] \quad \to \quad D\Lambda(m)^*[\mathrm{P}_{\Lambda(m) \leftarrow n} \xi_n^{(k+1)}]$$

- change $m = m^{(k)}$, $n = n^{(k)}$ during the iterations

The ℓ^2-TV Model

[Rudin, Osher, and Fatemi 1992; Lellmann, Strekalovskiy, Koetter, and Cremers 2013; Weinmann, Demaret, and Storath 2014]

For a manifold-valued image $f \in \mathcal{M}$, $\mathcal{M} = \mathcal{N}^{d_1,\ d_2}$, we compute

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{\alpha} F(p) + G(\Lambda(p)), \qquad \alpha > 0,$$

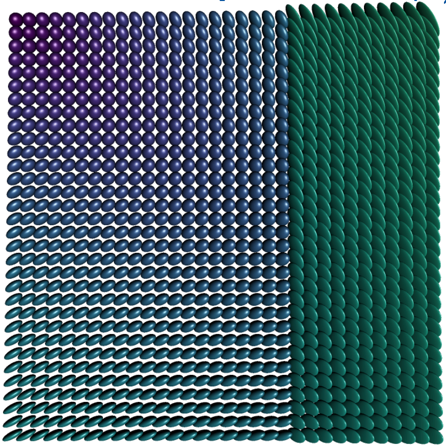
with

- data term $F(p) = \frac{1}{2} d^2_{\mathcal{M}}(p, f)$
- “forward differences” $\Lambda\colon \mathcal{M} \to (T\mathcal{M})^{d_1-1,\ d_2-1,\ 2}$,

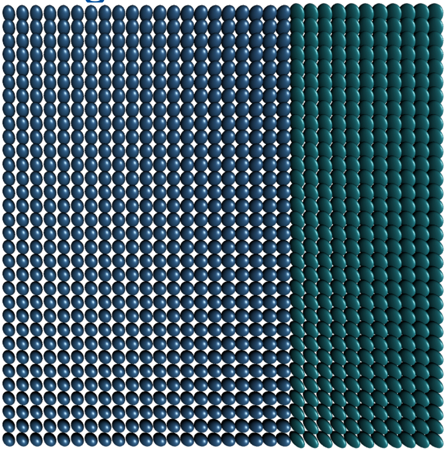
$$p \mapsto \Lambda(p) = \Big((\log_{p_i} p_{i+e_1},\ \log_{p_i} p_{i+e_2}) \Big)_{i \in \{1,\ldots,d_1-1\} \times \{1,\ldots,d_2-1\}}$$

- prior $G(X) = \|X\|_{g,q,1}$ similar to a collaborative TV [Duran, Moeller, Sbert, and Cremers 2016]

Numerical Example for a $\mathcal{P}(3)$-valued Image



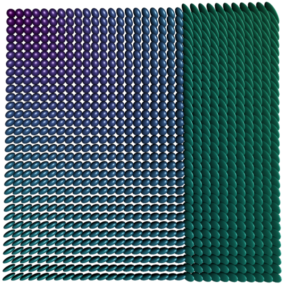
$\mathcal{P}(3)$-valued data.



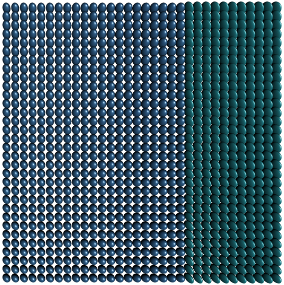
anisotropic TV, $\alpha = 6$.

- in each pixel we have a symmetric positive definite matrix
- Applications: denoising/inpainting e.g. of DT-MRI data

Numerical Example for a $\mathcal{P}(3)$-valued Image



$\mathcal{P}(3)$-valued data.

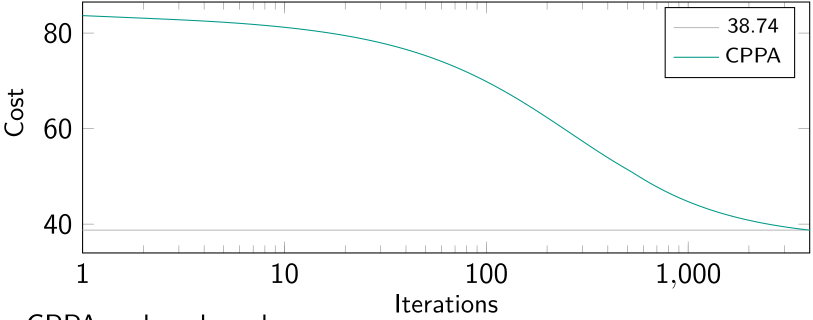


anisotropic TV, $\alpha = 6$.

Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$, $\beta = 0.93$	$\sigma = \tau = 0.4$, $\gamma = 0.2,\ m = I$
iterations	4000		
runtime	1235 s.		

Numerical Example for a $\mathcal{P}(3)$-valued Image



Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

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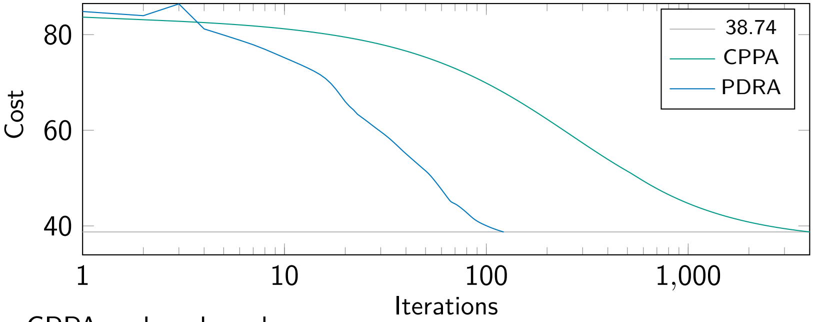
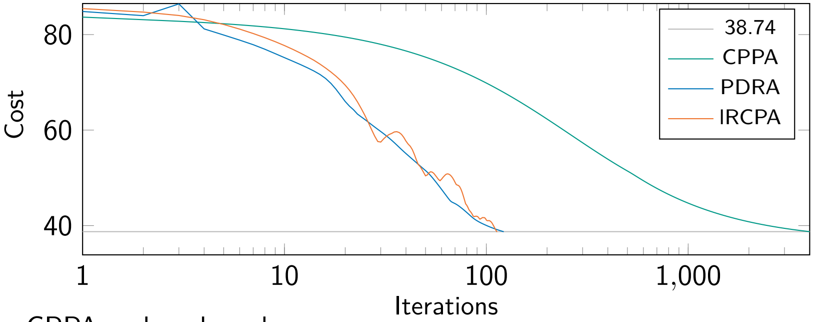


[Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

Approach. CPPA as benchmark

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$ $\beta = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000	122	
runtime	1235 s.	380 s.	

Numerical Example for a $\mathcal{P}(3)$-valued Image



[Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

Approach. CPPA as benchmark

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$ $\beta = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000	122	**113**
runtime	1235 s.	380 s.	**96.1 s.**

Summary & Outlook

Summary.

- We introduced a duality framework on Riemannian manifolds
- We derived a Riemannian Chambolle Pock Algorithm
- Numerical examples illustrate performance

Outlook.

- investigate $C(k)$ and the error of linearization
- strategies for choosing m, n (adaptively)
- alternative models of Fenchel duality (e. g. without m) [RB, Herzog, and Silva Louzeiro 2021]
- higher order methods non-smooth methods [Diepeveen and Lellmann 2021]

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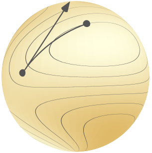
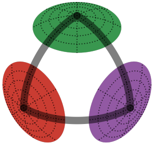
Reproducible Research

The algorithm is published in `Manopt.jl`, a Julia Package available at http://manoptjl.org.

It uses the interface from `ManifoldsBase.jl` and hence any manifold from `Manifolds.jl` can be used in the algorithms.
https://juliamanifolds.github.io/Manifolds.jl/
[Axen, Baran, RB, and Rzecki 2021]





Goal.

Being able to use an(y) algorithm for a(ny) model directly on a(ny) manifold easily and efficiently.

Alternatives.

- Manopt, manopt.org (Matlab, by N. Boumal)
- pymanopt, pymanopt.github.io (Python, by S. Weichwald, J. Townsend, N. Koep)

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