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Norwegian University of Science and Technology

Splitting Methods for Non-smooth Optimization on Manifolds

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joint work with
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Splitting Methods in Optimization

When solving an nonsmooth, high-dimensional optimisation problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} f(p)$$

we want to use that our $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ can be written as

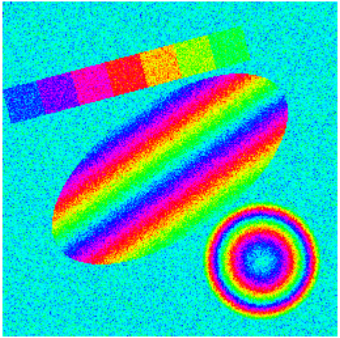
$$\operatorname*{arg\,min}_{p\in\mathcal{M}} \sum_{i=1}^{N} f_i(p)$$

for optimisation problems on a Riemannian manifold $\mathcal{M}$.

Manifold-valued Signal & Image Processing

Tasks in Image Processing is phrased as an optimisation problem.
Here. The pixel take values on a manifold

- phase-valued data ($\mathbb{S}^1$)
- wind-fields, GPS ($\mathbb{S}^2$)
- DT-MRI ($\mathcal{P}(3)$)
- EBSD, (grain) orientations ($\mathrm{SO}(n)$)



Artificial noisy phase-valued data.

Tasks. Denoising, Inpainting, labeling (classification), deblurring,...

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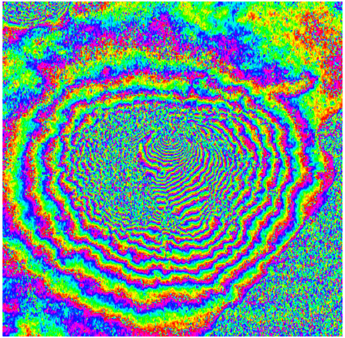

InSAR-Data of Mt. Vesuvius.
[Rocca, Prati, and Guarnieri 1997]

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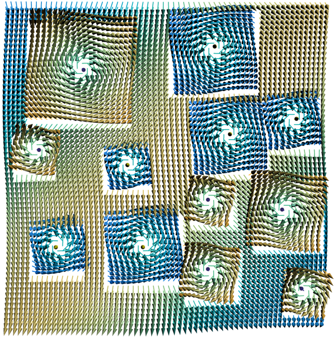

Artificial noisy data on the sphere $\mathbb{S}^2$.

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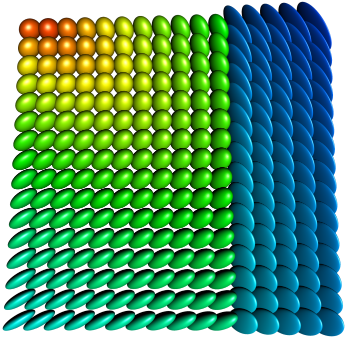

Artificial diffusion data,
each pixel is a symmetric positive matrix.

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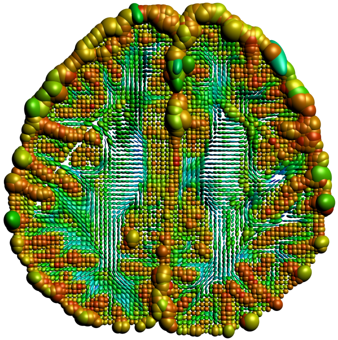

DT-MRI of the human brain.
Camino Profject: cmic.cs.ucl.ac.uk/camino

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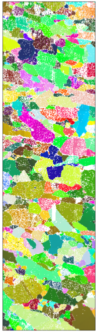

Grain orientations in EBSD data.
MTEX toolbox: mtex-toolbox.github.io

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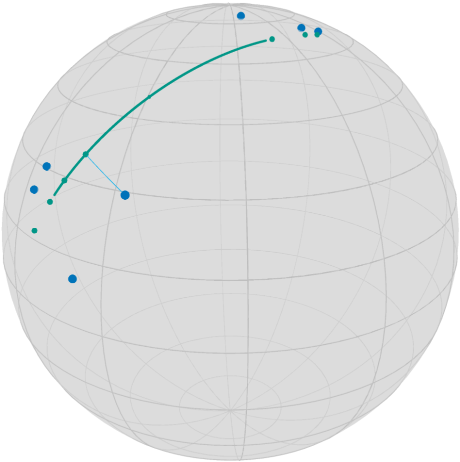
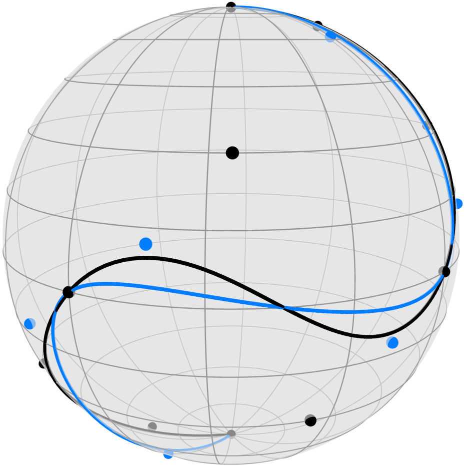
Regression & Interpolation



Regression. Find a geodesic/curve "explaining the data best"
[Rentmeesters 2011; Fletcher 2013]

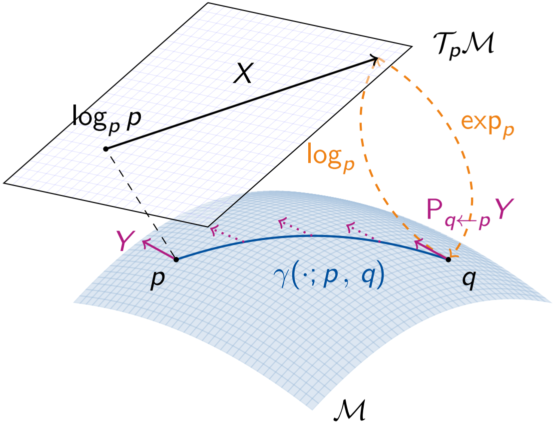


Interpolation. Interpolate data with a (Bézier) curve of min. acceleration.
[RB and Gousenbourger 2018]

A d-dimensional Riemannian manifold $\mathcal{M}$

Notation.

- Geodesic $\gamma(\cdot; p, q)$
- Tangent space $\mathcal{T}_p\mathcal{M}$
- inner product $(\cdot, \cdot)_p$
- Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
- Exponential map $\exp_p X = \gamma_{p,X}(1)$ where $\gamma_{p,X}(0) = p$ and $\dot\gamma_{p,X}(0) = X$
- Parallel transport $\mathrm{P}_{q\leftarrow p} Y$ "move" tangent vectors from $\mathcal{T}_p\mathcal{M}$ to $\mathcal{T}_q\mathcal{M}$



The Proximal Map

For $\varphi\colon \mathcal{M} \to (-\infty, +\infty]$ and $\lambda > 0$ the Proximum is defined by
[Moreau 1965; Rockafellar 1976; Ferreira and Oliveira 2002]

$$\operatorname{prox}_{\lambda\varphi}(p) := \operatorname*{arg\,min}_{q\in\mathcal{M}} \frac{1}{2} d_{\mathcal{M}}(q,p)^2 + \lambda\varphi(q).$$

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$$p_{k+1} = \operatorname{prox}_{\lambda_k\varphi}(p_k)$$

converges (weakly) if $\{\lambda_k\}_k \notin \ell_1(\mathbb{N})$ on Hadamard manifolds.

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But. computing one step (numerically) might be quite expensive.

Cyclic Proximal Point Algorithm

Idea. Split $f = \sum_{i=1}^{N} f_i$ and apply the Cyclic Proximal Point-Algorithmus (CPPA):

[Bertsekas 2011; Bačák 2014]

$$p_{k+\frac{i+1}{N}} = \operatorname{prox}_{\lambda_k f_i}\Bigl(p_{k+\frac{i}{N}}\Bigr), \quad i = 0, \ldots, N-1, \; k = 0, 1, \ldots$$

This converges on to a minimizer of f on a Hadamard manifold $\mathcal{M}$ if

- all f_i proper, convex, lsc.
- $\{\lambda_k\}_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})\backslash\ell_1(\mathbb{N})$.

Applications of CPPA

The algorithm works well also

- with inexact/approximate evaluations of the proximal maps
  [Bačák, RB, Steidl, and Weinmann 2016]
- works numerically on non-Hadamard manifolds
- a lot of simple proximal maps available in closed form:
  1. distance $\varphi(p) = \frac{1}{n} d_{\mathcal{M}}(p,q)^n, n \in \{1,2\},\ p \in \mathcal{M}$
  2. finite difference $\varphi(p) = \frac{1}{n} d_{\mathcal{M}}(p_1,p_2)^n, n \in \{1,2\},\ p \in \mathcal{M}^2$
- second order difference $\varphi(p) = d_{2,\mathcal{M}}(p_1,p_2,p_3),\ p \in M^3$

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Example. ℓ^2-TV for a given signal $f \in \mathcal{M}^N$

$$\operatorname*{arg\,min}_{p \in \mathcal{M}^N} \frac{1}{2} d_{\mathcal{M}^N}(f,p)^2 + \sum_{i=1}^{N-1} d_{\mathcal{M}}(p_i, pi+1)$$

The Reflection

A map $\mathcal{R}_p$ is called Reflection on $\mathcal{M}$, if

$$\mathcal{R}_p(p) = p \quad \text{and} \quad D_p\mathcal{R}_p = -I \quad \text{hold.}$$

Analogously: Reflection at the prox we denote by

$$\mathcal{R}_{\lambda\varphi}(x) = \mathcal{R}_{\operatorname{prox}_{\lambda\varphi}(x)}(x)$$

Example. On $\mathbb{R}^n$ we have $\mathcal{R}_p(x) = 2p - x = p - (x - p)$.

The Douglas-Rachford Algorithm (DRA)

Goal. Find a minimizer of two proper, convex, lsc. functions

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} F(p) + G(p)$$

Iteration: For $p_0 \in \mathcal{M}$ compute the Krasnoselskii-Mann-iteration, i.e.,

[RB, Persch, and Steidl 2016]

$$q_k = \mathcal{R}_{\lambda F}(\mathcal{R}_{\lambda G}(p_k))$$
$$p_{k+1} = \gamma\big(\beta_k; p_k\,,\ q_k\big)$$

with $\beta_k \in (0,1)$ and $\sum_{k\in\mathbb{N}} \beta_k(1-\beta_k) = \infty$

Convergence of the Douglas–Rachford Algorithm

Theorem [Kakavandi 2013]

Let $\mathcal{R}_{\lambda F}$,$\mathcal{R}_{\lambda G}$ be non-expansive and hence $T = \mathcal{R}_{\lambda F} \circ \mathcal{R}_{\lambda G}$ is nonexpansive. Let T possess a fix point $\hat{q}$.
Then the sequence $\{q_k\}$ in the DRA converges for every start point $p_0 \in \mathcal{M}$ to a fix point $\hat{q}$ of T (in the q_k).

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Theorem
[RB, Persch, and Steidl 2016]

Let F, G be proper, convex, lsc., let there be a minimizer $p^\star$ of $F + G$, and let $T = \mathcal{R}_{\lambda F} \circ \mathcal{R}_{\lambda G}$ be non-expansive.
Then there exists for every $p^\star$ a fix point $\hat{q}$ of T, such that

$$p^\star = \operatorname{prox}_{\lambda\psi}(\hat{q})$$

holds. Further, for every $\hat{q}$, the point $\operatorname{prox}_{\lambda\psi}(\hat{q})$ is a minimizer of $F + G$.

Imaging: Parallel (or consensus) Douglas–Rachford.

For a sum $f = \sum_{i=1}^{N} f_i$ vectorize the objective

$$G(\boldsymbol{x}) = \sum_{i=1}^{N} f_i(x_i), \qquad \boldsymbol{x} \in \mathcal{M}^N$$

$\Rightarrow$ $\operatorname{prox}_{\lambda G}$ is element-wise easy proxes

And

$$F(\boldsymbol{x}) = \iota_D(\boldsymbol{x}), \qquad D := \left\{\boldsymbol{x} \in \mathcal{M}^N : x_1 = x_2 = \cdots = x_N\right\}$$

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We obtain convergence on Hadamard manifolds of constant curvature and numerically works fine on Hadamard manifolds.

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The Riemannian $m-$Fenchel Conjugate

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

alternative approaches: [Ahmadi Kakavandi and Amini 2010; RB, Herzog, and Silva Louzeiro 2021]

Let $m \in \mathcal{C} \subset \mathcal{M}$ be given and $F\colon \mathcal{C} \to \overline{\mathbb{R}}$.

The m-Fenchel conjugate $F^*_m\colon \mathcal{T}^*_m\mathcal{M} \to \overline{\mathbb{R}}$ is defined by

$$F^*_m(\xi_m) := \sup_{X \in \mathcal{L}_{\mathcal{C},m}} \bigl\{ \langle \xi_m , X \rangle - F(\exp_m X) \bigr\},$$

where $\mathcal{L}_{\mathcal{C},m} := \{ X \in \mathcal{T}_m\mathcal{M} \mid q = \exp_m X \in \mathcal{C} \text{ and } \|X\|_p = d(q,p) \}$.

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A new model. This can be used to minimize

$$\operatorname*{arg\,min}_{p \in \mathcal{C}} F(p) + G(\Lambda(p))$$

where we have to linearize $\Lambda\colon \mathcal{M} \to \mathcal{N}$ as $\Lambda(p) \approx \exp_{\Lambda(m)} D\Lambda(m)[\log_m p]$

[Valkonen 2014]

The Exact Riemannian Chambolle–Pock Algorithm (eRCPA)

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021; Chambolle and Pock 2011]

Goal. Minimize $F(p) + G(\Lambda(p))$ with an arbitrary map $\Lambda \mathcal{M} \to \mathcal{N}$.

Input: $p^{(0)} \in \mathbb{R}^d$, $\xi^{(0)} \in \mathbb{R}^d$, and parameters σ, τ, $\theta > 0$

1: $k \leftarrow 0$
2: $\bar{p}^{(0)} \leftarrow p^{(0)}$
3: **while** not converged **do**
4: $\quad \xi^{(k+1)} \leftarrow \operatorname{prox}_{\tau G^*}\bigl(\xi^{(k)} + \tau(\quad \Lambda(\bar{p}^{(k)})) \bigr)$
5: $\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\Bigl(p^{(k)} \quad \bigl(-\sigma \ \Lambda \quad {}^* \xi^{(k+1)}\bigr)^\sharp \Bigr)$
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The Exact Riemannian Chambolle–Pock Algorithm (eRCPA)

[RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021; Chambolle and Pock 2011]

Goal. Minimize $F(p) + G(\Lambda(p))$ with an arbitrary map $\Lambda\mathcal{M} \to \mathcal{N}$.

Input: m, $p^{(0)} \in \mathcal{C} \subset \mathcal{M}$, $n = \Lambda(m)$, $\xi_n^{(0)} \in \mathcal{T}_n^*\mathcal{N}$, and parameters $\sigma, \tau, \theta > 0$

$k \leftarrow 0$

$\bar{p}^{(0)} \leftarrow p^{(0)}$

while not converged **do**

$\quad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau G_n^*}\Bigl(\xi_n^{(k)} + \tau\bigl(\log_n \Lambda(\bar{p}^{(k)})\bigr)^\flat\Bigr)$

$\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\Bigl(p^{(k)} + \mathrm{P}_{p^{(k)} \leftarrow m}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)$

$\quad \bar{p}^{(k+1)} \leftarrow p^{(k+1)} + \theta(p^{(k+1)} - p^{(k)})$

$\quad k \leftarrow k + 1$

end while

Output: $p^{(k)}$

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Input: m, $p^{(0)} \in \mathcal{C} \subset \mathcal{M}$, $n = \Lambda(m)$, $\xi_n^{(0)} \in \mathcal{T}_n^*\mathcal{N}$, and parameters σ, τ, $\theta > 0$

1: $k \leftarrow 0$
2: $\bar{p}^{(0)} \leftarrow p^{(0)}$
3: **while** not converged **do**
4: $\quad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau G_n^*}\Bigl(\xi_n^{(k)} + \tau\bigl(\log_n \Lambda(\bar{p}^{(k)})\bigr)^\flat\Bigr)$
5: $\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\Bigl(\exp_{p^{(k)}}\Bigl(\mathrm{P}_{p^{(k)}\leftarrow m}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)\Bigr)$
6: $\quad \bar{p}^{(k+1)} \leftarrow p^{(k+1)} + \theta(p^{(k+1)} - p^{(k)})$
7: $\quad k \leftarrow k + 1$
8: **end while**

Output: $p^{(k)}$

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while not converged **do**

$\quad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau G_n^*}\Bigl(\xi_n^{(k)} + \tau\bigl(\log_n \Lambda(\bar{p}^{(k)})\bigr)^\flat\Bigr)$

$\quad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma F}\biggl(\exp_{p^{(k)}}\Bigl(\mathrm{P}_{p^{(k)}\leftarrow m}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)\biggr)$

$\quad \bar{p}^{(k+1)} \leftarrow p^{(k+1)} - \theta(p^{(k)} - p^{(k+1)})$

$\quad k \leftarrow k + 1$

end while

Output: $p^{(k)}$

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$\quad \bar{p}^{(k+1)} \leftarrow \exp_{p^{(k+1)}}\bigl(-\theta \log_{p^{(k+1)}} p^{(k)}\bigr)$

$\quad k \leftarrow k + 1$

end while

Output: $p^{(k)}$

Manifolds.jl & Manopt.jl

[RB 2022]

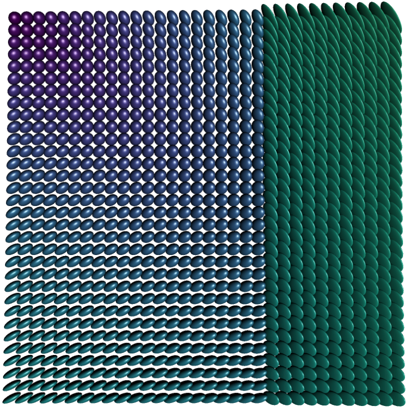
The presented algorithms are implemented within the Julia package **Manopt.jl**. The Julia package provides general framework to implement optimisation algorithms on Manifolds (similar to Manopt, pymanopt)

[Boumal, Mishra, Absil, and Sepulchre 2014; Townsend, Koep, and Weichwald 2016]

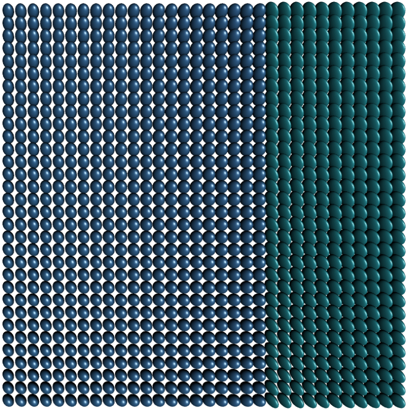
The algorithms are implemented using on **ManifoldsBase.jl**, which is an interface for manifolds. A corresponding Library of manifolds is provided in **Manifolds.jl**. [Axen, Baran, RB, and Rzecki 2021]

Motivation.
Provide an efficient, well-tested, well-documented Library of Riemannian manifolds.

Numerical Example for a $\mathcal{P}(3)$-valued Image



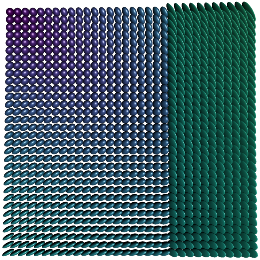
$\mathcal{P}(3)$-valued data.



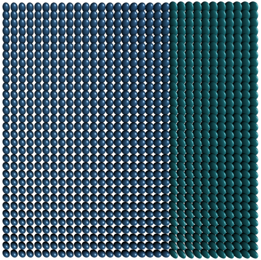
anisotropic TV, $\alpha = 6$.

- in each pixel we have a symmetric positive definite matrix
- Applications: denoising/inpainting e.g. of DT-MRI data

Numerical Example for a $\mathcal{P}(3)$-valued Image



$\mathcal{P}(3)$-valued data.

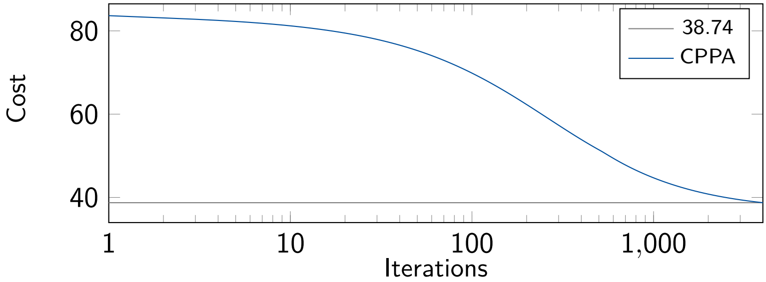


anisotropic TV, $\alpha = 6$.

Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$ $\beta = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000		
runtime	1235 s.		

Numerical Example for a $\mathcal{P}(3)$-valued Image



Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

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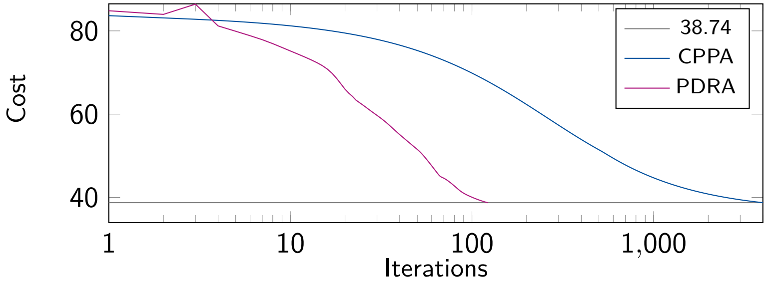
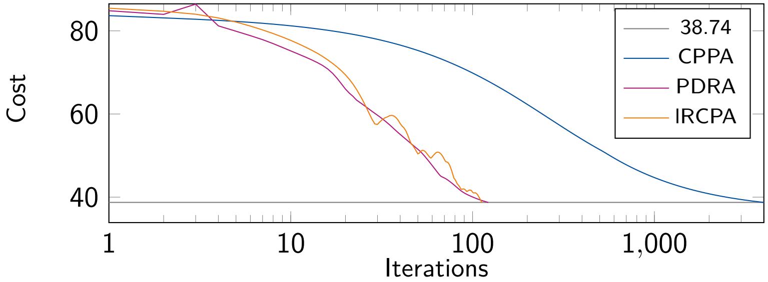


Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$ $\beta = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000	122	
runtime	1235 s.	380 s.	

Numerical Example for a $\mathcal{P}(3)$-valued Image



Approach. CPPA as benchmark [Bačák 2014; RB, Herzog, Silva Louzeiro, Tenbrinck, and Vidal-Núñez 2021]

	CPPA	PDRA	IRCPA
parameters	$\lambda_k = \frac{4}{k}$	$\lambda = 0.58$ $\beta = 0.93$	$\sigma = \tau = 0.4$ $\gamma = 0.2,\ m = I$
iterations	4000	122	**113**
runtime	1235 s.	380 s.	**96.1 s.**

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